SOLVABILITY AND LIMIT BICHARACTERISTICS

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1. Introduction

We shall consider the solvability for a classical pseudodifferential operator P on a C^∞ manifold X. This means that P has an expansion $p_m + p_{m-1} + \dots$ where $p_k \in S^k_{hom}$ is homogeneous of degree k and $p_m = \sigma(P)$ is the principal symbol of the operator. The operator P is solvable at a compact set $K \subseteq X$ if the equation

$$Pu = v \tag{1.1}$$

has a local solution $u \in \mathcal{D}'(X)$ in a neighborhood of K for any $v \in C^\infty(X)$ in a set of finite codimension. We can also define the microlocal solvability at any compactly based cone $K \subset T^*X$, see Definition 2.8.

A pseudodifferential operator is of principal type if the Hamilton vector field

$$H_p = \sum_{j=1}^n \partial_{\xi_j} p \partial_{x_j} - \partial_{x_j} p \partial_{\xi_j} \tag{1.2}$$

of the principal symbol p does not have the radial direction $\langle \xi, \partial_\xi \rangle$ at $p^{-1}(0)$, in particular $H_p \neq 0$ then. By homogeneity H_p is well defined on the cosphere bundle $S^*X = \{\, (x,\xi) \in T^*X : \ |\xi| = 1 \,\}$, defined by some choice of Riemannean metric. For pseudodifferential operators of principal type, it is known [1] [3] that local solvability at a point is equivalent to condition (Ψ) which means that

$$\text{(1.3)} \quad \operatorname{Im}(ap) \text{ does not change sign from } - \text{ to } + \\ \text{along the oriented bicharacteristics of } \operatorname{Re}(ap)$$

for any $0 \neq a \in C^\infty(T^*M)$. Oriented bicharacteristics are the positive flow-outs of the Hamilton vector field $H_{\operatorname{Re}(ap)} \neq 0$ on $\operatorname{Re}(ap) = 0$. Bicharacteristics of $\operatorname{Re} ap$ are also called semi-bicharacteristics of p.

We shall consider the case when the principal symbol is real and vanishes of at least second order at an involutive manifold Σ_2, thus P is not of principal type. For operators which are not of principal type, the values of the subprincipal symbol p_{m-1} at Σ_2 becomes important. In the case when principal symbol $p = \xi_1\xi_2$, Mendoza and Uhlman [5] proved that P was not solvable if the subprincipal symbol changed sign on the x_1 or x_2 lines when $\xi_1 = \xi_2 = 0$. Mendoza [6] generalized this to the case when the principal symbol vanishes on an involutive submanifold having an indefinite Hessian with rank equal to the

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codimension of the manifold. The Hessian then gives well-defined limit bicharacteristis over the submanifold, and P is not solvable if the subprincipal symbol changes sign on any of these limit bicharacteristics. This corresponds to condition (P) on the limit bicharacteristics (no sign changes) because in this case one gets both directions when taking the limits.

In this paper, we shall extend this result to more general pseudodifferential operators. As in the previous cases, the operator will have real principal symbol and we shall consider the limits of bicharacteristics at the set where the principal symbol vanishes of at least second order. The convergence shall be as smooth curves, then the limit bicharacteristic also is a smooth curve. We shall also need uniform bounds on the curvature of the characteristics at the bicharacteristics, but only along the tangent space of a Lagrangean submanifold of the characteristics, which we call a grazing Lagrangean space, see (2.5). This gives uniform bounds on the linearization of the normalized Hamilton flow on the tangent space of this submanifold at the bicharacteristics. Our main result is Theorem 2.9, which essentially says that under these conditions the operator is not solvable at the limit bicharacteristic if the quotient of the imaginary part of the subprincipal symbol with the norm of the Hamilton vector field switches sign from $-$ to $+$ on the bicharacteristics and becomes unbounded as they converge to the limit bicharacteristic.

2. Statement of results

Let the principal symbol p be real valued, $\Sigma = p^{-1}(0)$ be the characteristics, and Σ_2 be the set of double characteristics, i.e., the points on Σ where $dp = 0$. Let $\{ \Gamma_j \}_{j=1}^{\infty}$ be a set of bicharacteristics of p on S^*X so that $\Gamma_j \subset \Sigma \setminus \Sigma_2$ are uniformly bounded in C^∞ when parametrized on a uniformly bounded interval (for example with respect to the arc length). These bounds are defined with respect to some choice of Riemannean metric on T^*X, but different choices of metric will only change the bounds with fixed constants. In particular, we have a uniform bound on the arc lengths:

$$|\Gamma_j| \le C \qquad \forall\, j \tag{2.1}$$

We also have that $\Gamma_j = \{ \gamma_j(t) : \ t \in I_j \}$ with $|\gamma_j'(t)| \equiv 1$ and $|I_j| \le C$, then $|\gamma_j^{(k)}(t)| \le C_k$ for $t \in I_j$ and all j, $k \ge 1$. Let $\widetilde{p} = p/|\nabla p|$ then the normalized Hamilton vector field is equal to

$$H_{\widetilde{p}} = |H_p|^{-1} H_p \qquad \text{on } p^{-1}(0) \setminus \Sigma_2$$

then Γ_j is uniformly bounded in C^∞ if

$$|H_{\widetilde{p}}^k \nabla \widetilde{p}| \le C_k \qquad \text{on } \Gamma_j \quad \forall\, j, k \tag{2.2}$$

where $\nabla\widetilde{p}$ is the gradient of $\widetilde{p}$. Thus the normalized Hamilton vector field $H_{\widetilde{p}}$ is uniformly bounded in C^∞ as a vector field over Γ. Observe that the bicharacteristics have a natural orientation given by the Hamilton vector field. Now the set of bicharacteristic curves $\{ \Gamma_j \}_{j=1}^{\infty}$ is uniformly bounded in C^∞ when parametrized with respect to the arc length,

and therefore it is a precompact set. Thus there exists a subsequence Γ_{j_k}, $k \to \infty$, that converge to a smooth curve Γ (possibly a point), called a limit bicharacteristic by the following definition.

Definition 2.1. We say that a sequence of smooth curves Γ_j on a smooth manifold converges to a smooth limit curve Γ (possibly a point) if there exist parametrizations on uniformly bounded intervals that converge in C^∞. If $p \in C^\infty(T^*X)$, then a smooth curve $\Gamma \subset \Sigma_2 \bigcap S^*X$ is a *limit bicharacteristic* of p if there exists bicharacteristics Γ_j that converge to it.

Naturally, this definition is invariant, and the set $\{\, \Gamma_j \,\}_{j=1}^\infty$ may have subsequences converging to several different limit bicharacteristics, which could be points. In fact, if Γ_j is parametrized with respect to the arc length on intervals I_j such that $|I_j| \to 0$, then we find that Γ_j converges to a limit curve which is a point. Observe that if Γ_j converge to a limit bicharacteristic Γ, then (2.1) and (2.2) hold for Γ_j.

Example 2.2. Let Γ_j be the curve parametrized by

$$[0,1] \ni t \mapsto \gamma_j(t) = (\cos(jt), \sin(jt))/j$$

Since $|\gamma_j'(t)| = 1$ the curves are parametrized with respect to arc length, and we have that $\Gamma_j \to (0,0)$ in C^0, but not in C^∞ since $|\gamma_j''(t)| = j$. If we parametrize Γ_j with $x = jt \in [0,j]$ we find that Γ_j converge to $(0,0)$ in C^∞ but not on uniformly bounded intervals.

Example 2.3. Let P have real principal symbol $p = w_1^k - a(w')$ in the coordinates $(w_1, w') \in T^*\mathbf{R}^n \setminus 0$, where $w_1 = a(w') = 0$ at Σ_2 and $k \geq 2$. This case includes the cases where the operator is microhyperbolic, then $k = 2$ and $a(w')$ vanishes of exactly second order at Σ_2. If k is even then in order to have limit bicharacteristics, it is necessary that $a \not\leq 0$ in a neighborhood of Σ_2, and then Σ is given locally by $w_1 = \pm a(w')^{1/k}$. We find that (2.2) is satisfied if $H_{\tilde{p}}^j w_1 = 0$ and $H_{\tilde{p}}^j \nabla a = \mathcal{O}(|\nabla a|)$ for any j when $w_1 = 0$.

But we shall also need a condition on the differential of the Hamilton vector field H_p at the bicharacteristics along a Lagrangean space, which will give bounds on the curvature of the characteristics in these directions. In the following, a section of Lagrangean spaces L over a bicharacteristic Γ will be a map

$$\Gamma \ni w \mapsto L(w) \subset T_w(T^*X)$$

such that $L(w)$ is a Lagrangean subspace in $T_w\Sigma$, $\forall\, w \in \Gamma$, where $\Sigma = p^{-1}(0)$. If the section L is C^1 then it has tangent space $TL \subset T_L(T_\Gamma(T^*X))$. Since the Hamilton vector field $H_p(w)$ is in $T_w\Gamma \subset T_w(T^*(X))$, we find the linearization (or first order jet) of H_p at $w \in \Gamma$ is in $T_{H_p(w)}(T_w(T^*X))$.

Definition 2.4. For a bicharacteristic Γ of p we say that a C^1 section of Lagrangean spaces L over Γ is a section of *grazing Lagrangean spaces* of Γ if $L \subset T\Sigma$ and the linearization (or first order jet) of H_p is in TL at Γ.

Observe that since $L(w) \subset T_w\Sigma$ is Lagrangean we find $dp(w)\big|_{L(w)} = 0$ and $H_p(w) \in L(w)$ when $w \in \Gamma$. The linearization of $H_p(w)$ is given by the second order Taylor expansion of p at w and since $L(w)$ is Lagrangean we find that terms in that expansion that vanish on $L(w)$ have Hamilton field parallel to L. Thus, the condition that the linearization of $H_p(w)$ is in $TL(w)$ only depends on the restriction to $L(w)$ of the second order Taylor expansion of p at w. Since $L(w) \subset T_w\Sigma$, we find that Definition 2.4 is invariant under multiplication of p by non-vanishing factors because $p(w) = dp(w)\big|_{L(w)} = 0$. Thus we can replace $H_p(w)$ by the normalized Hamilton field $H_{\widetilde{p}}$ in the definition.

Example 2.5. Let L be given by $\Gamma \ni w \mapsto (w, L(w))$ with coordinates chosen so that $L(w) = \{\, (x,\xi) \in \mathbf{R}^{2n} :\ \xi = A(w)x \,\}$ where $A(w) : \mathbf{R}^n \mapsto \mathbf{R}^n$ is linear. Then we can parametrize the tangent space of $L(w_0)$ with

$$L'(w_0) :\ T_{w_0}\Gamma \times L(w_0) \ni (\delta w, \delta z) \mapsto (w_0, L(w_0), \delta w, (\delta x, A'(w_0)(\delta w)\delta x) + \delta z)$$

and L is C^1 if $A \in C^1(\Gamma, \mathbf{R}^n)$. In these coordinates, the linearization of $H_p(w_0)$ is given by

$$H_p'(w_0) :\ T_{H_p(w_0)}(T_{w_0}(T^*X)) \ni (\delta x, \delta\xi, \delta z, \delta\zeta) \mapsto H_p(w_0)(\delta z, \delta\zeta) + \partial_{x,\xi} H_p(w_0)(\delta x, \delta\xi)$$

where $H_p(w_0)(\delta z, \delta\zeta) = \partial_\xi p(w_0)\delta z - \partial_x p(w_0)\delta\zeta \in L(w_0)$.

By Definition 2.4 we find that the linearization of H_p gives an evolution equation for the section L, see Example 2.6. Choosing a Lagrangean subspace of $T_{w_0}\Sigma$ at $w_0 \in \Gamma$ then determines L along Γ, so L must be smooth. Actually, L is the tangent space at Γ of a Lagrangean submanifold of Σ, see (4.22).

Example 2.6. Let $p = \tau - (\langle A(t)x, x\rangle + 2\langle B(t)x, \xi\rangle + \langle C(t)\xi, \xi\rangle)/2$ where $(x,\xi) \in T^*\mathbf{R}^n$, $A(t)$, $B(t)$ and $C(t) \in C^\infty$ are real $n \times n$ matrices, where $A(t)$ and $C(t)$ are symmetric, and let $\Gamma = \{\, (t, 0, 0, \xi_0) :\ t \in I \,\}$. Then $H_p = \partial_t$ at Γ,

$$p^{-1}(0) = \{\, \tau = \langle A(t)x, x\rangle/2 + \langle B(t)x, \xi\rangle + \langle C(t)\xi, \xi\rangle/2 \,\}$$

and the linearization of the Hamilton field H_p at $(t, 0, 0, \xi_0)$ is

$$\partial_t + \langle A(t)y + B^*(t)\eta, \partial_\eta\rangle - \langle B(t)y + C(t)\eta, \partial_y\rangle \qquad (y,\eta) \in T^*\mathbf{R}^n \tag{2.3}$$

The linearization of the normalized Hamilton field $H_{\widetilde{p}}$ is the same as (2.3), since $|H_p| \cong 1$ modulo quadratic terms in (y, η). Since $dp = d\tau$ at Γ, a C^1 section of Lagrangean spaces in $T_\Gamma\Sigma$ is for example given by

$$L(t) = \{\, (s, y, 0, E(t)y) : (s, y) \in \mathbf{R}^n \,\}$$

where $E(t) \in C^1$ is symmetric with $E(0) = 0$, and by choosing linear symplectic coordinates (y, η) we can obtain any such section on this form. By applying (2.3) on $\eta - E(t)y$, which vanishes on $L(t)$, we obtain that $L(t)$ is a grazing Lagrangean space if

$$E'(t) = A(t) + 2\operatorname{Re} B(t)E(t) + E(t)C(t)E(t) \tag{2.4}$$

with $\operatorname{Re} F = (F + F^*)/2$, see (4.20). Then by uniqueness $L(t)$ is constant in t if and only if $A(t) \equiv 0$, observe that then $A(t) = \operatorname{Hess} p\big|_{L(t)}$. In general, $\operatorname{Hess} p\big|_{L(t)}$ is given by the right hand side of (2.4).

Observe that we may choose symplectic coordinates $(t, x; \tau, \xi)$ so that $\tau = p$ and the fiber of $L(w)$ is equal to $\{\, (s, y, 0, 0) : (s, y) \in \mathbf{R}^n \,\}$ at $w \in \Gamma = \{\, (t, 0; 0, \xi_0) : \ t \in I \,\}$. But it is not clear that we can do that *uniformly* for a family of bicharacteristics $\{\, \Gamma_j \,\}$, for that we need an additional condition. Now we assume that there exists a grazing Lagrangean space L_j of Γ_j, $\forall\, j$, such that the normalized Hamilton vector field $H_{\widetilde{p}}$ satisfies

$$\left| dH_{\widetilde{p}}(w)\big|_{L_j(w)} \right| \le C \qquad \text{for } w \in \Gamma_j \quad \forall\, j \tag{2.5}$$

Since the mapping $\Gamma_j \ni w \mapsto L_j(w)$ is determined by the linearization of $H_{\widetilde{p}}$ on L_j, thus by $dH_{\widetilde{p}}(w)\big|_{L_j(w)}$, condition (2.5) implies that $\Gamma_j \ni w \mapsto L_j(w)$ is uniformly in C^1, see Example 2.6. Observe that condition (2.2) gives (2.5) in the direction of $T_w\Gamma_j \subset L_j(w)$. Clearly condition (2.5) is invariant under changes of symplectic coordinates and multiplications with non-vanishing real factors. Now if $0 \ne u \in C^\infty$ has values in $\mathbf{R}^n$ and $\omega = u/|u|$ is the normalization, then

$$\partial\omega = \partial u/|u| - \langle u, \partial u\rangle u/|u|^3$$

This gives that (2.5) is equivalent to

$$\left| \Pi d\nabla p\big|_{L_j} \right| \le C|\nabla p| \qquad \text{on } \Gamma_j \quad \forall\, j \tag{2.6}$$

where $\Pi v = v - \langle v, \nabla p\rangle \nabla p/|\nabla p|^2$ is the orthogonal projection along ∇p. Condition (2.6) gives a uniform bound on the curvature of level surface $p^{-1}(0)$ in the directions given by L_j over Γ_j. Observe that the invariance of condition (2.6) can be checked directly since $d(ap) = adp$ and $d^2(ap) = ad^2p + dadp = ad^2p$ on L_j over Γ_j.

The invariant subprincipal symbol p_s will be important for the solvability of the operator. For the usual Kohn-Nirenberg quantization of pseudodifferential operators, the subprincipal symbol is equal to

$$p_s = p_{m-1} - \frac{1}{2i}\sum_j \partial_{\xi_j}\partial_{x_j} p \tag{2.7}$$

and for the Weyl quantization it is p_{m-1}.

Now for the principal symbol we shall denote

$$0 < \min_{\Gamma_j} |H_p| = \kappa_j \to 0 \qquad j \to \infty \tag{2.8}$$

and for the subprincipal symbol p_s we shall assume the following condition

$$\min_{\partial\Gamma_j} \int \operatorname{Im} p_s |H_p|^{-1}\, ds/|\log \kappa_j| \to \infty \qquad j \to \infty \tag{2.9}$$

where the integration is along the natural orientation given by H_p on Γ_j starting at some point $w_j \in \mathring{\Gamma}_j$. This means that the integral must have a minimum less or equal to zero in the interior of Γ_j, and starting the integration at that minimum can only improve (2.9). Thus we may assume that the integral in (2.9) is non-negative. Observe that if (2.9) holds then there must be a change of sign of $\operatorname{Im} p_s$ from $-$ to $+$ on Γ_j, and

$$\max_{\Gamma_j} (-1)^{\pm 1} \operatorname{Im} p_s/|H_p||\log \kappa_j| \to \infty \qquad j \to \infty \tag{2.10}$$

for both signs. Observe that condition (2.9) is invariant under symplectic changes of coordinates and multiplication with elliptic pseudodifferential operators, thus under conjugation with elliptic Fourier integral operators. In fact, multiplication only changes the subprincipal symbol with the same non-vanishing factors as $|H_p|$ and terms proportional to $|\nabla p| = |H_p|$. Now by choosing symplectic coordinates $(t, x; \tau, \xi)$ near a given point $w_0 \in \Gamma_j$ so that $p = \alpha\tau$ near w_0 with $\alpha \neq 0$, we obtain that $\partial_{x_j}\partial_{\xi_j} p = 0$ at Γ_j and $\partial_t\partial_\tau p = \partial_t \alpha = \partial_t |\nabla p|$ at Γ_j near w_0. Thus, the second term in (2.7) only gives terms in condition (2.9) which are bounded by

$$\int \partial_t |\nabla p|/|\nabla p|\, ds/|\log(\kappa_j)| \lesssim |\log(|\nabla p|)|/|\log(\kappa_j)| \lesssim 1 \qquad j \gg 1 \tag{2.11}$$

since $\kappa_j \le |\nabla p|$ on Γ_j. Here $a \lesssim b$ (and $b \gtrsim a$) means that $a \le Cb$ for some $C > 0$. Thus we obtain the following remark.

Remark 2.7. We may replace the subprincipal symbol p_s by p_{m-1} in (2.9), since the difference is bounded as $j \to \infty$.

We shall study the microlocal solvability, which is given by the following definition. Recall that $H^{loc}_{(s)}(X)$ is the set of distributions that are locally in the L^2 Sobolev space $H_{(s)}(X)$.

Definition 2.8. If $K \subset S^*X$ is a compact set, then we say that P is microlocally solvable at K if there exists an integer N so that for every $f \in H^{loc}_{(N)}(X)$ there exists $u \in \mathcal{D}'(X)$ such that $K \bigcap \operatorname{WF}(Pu - f) = \emptyset$.

Observe that solvability at a compact set $M \subset X$ is equivalent to solvability at $S^*X\big|_M$ by [4, Theorem 26.4.2], and that solvability at a set implies solvability at a subset. Also, by Proposition 26.4.4 in [4] the microlocal solvability is invariant under conjugation by elliptic Fourier integral operators and multiplication by elliptic pseudodifferential operators. The following is the main result of the paper.

Theorem 2.9. *Let $P \in \Psi^m_{cl}(X)$ have real principal symbol $\sigma(P) = p$, and subprincipal symbol p_s. Let $\{\, \Gamma_j \,\}_{j=1}^\infty$ be a family of bicharacteristic intervals of p in S^*X so that* (2.5) *and* (2.9) *hold. Then P is not microlocally solvable at any limit bicharacteristics of $\{\, \Gamma_j \,\}_j$.*

To prove Theorem 2.9 we shall use the following result. Let $\|u\|_{(k)}$ be the L^2 Sobolev norm of order k for $u \in C_0^\infty$ and P^* the L^2 adjoint of P.

Remark 2.10. If P is microlocally solvable at $\Gamma \subset S^*X$, then Lemma 26.4.5 in [4] gives that for any $Y \Subset X$ such that $\Gamma \subset S^*Y$ there exists an integer ν and a pseudodifferential operator A so that $\mathrm{WF}(A) \cap \Gamma = \emptyset$ and

$$\|u\|_{(-N)} \le C(\|P^*u\|_{(\nu)} + \|u\|_{(-N-n)} + \|Au\|_{(0)}) \qquad u \in C_0^\infty(Y) \tag{2.12}$$

where N is given by Definition 2.8. Since that definition also holds for larger N, we may take $\nu = N \ge 0$.

We shall use Remark 2.10 in Section 7 to prove Theorem 2.9 by constructing approximate local solutions to $P^*u = 0$. We shall first prepare and get a microlocal normal form for the adjoint operator, which will be done in Section 4. Then we shall solve the eikonal equation in Section 5 and the transport equations in Section 6.

3. Examples

Example 3.1. Let P have principal symbol $p = \prod_j p_j$ which is a product of real symbols p_j of principal type, such that $p_j = 0$ on Σ_2, $\forall\, j$, and $p_j \ne p_k$ on $\Sigma \setminus \Sigma_2$ when $j \ne k$. We find that if $\Gamma \subset p_k^{-1}(0)$ then

$$|H_p| = |H_{p_k}| \prod_{j \ne k} |p_j| = |H_{p_k}| q_k$$

where $q_k > 0$ on $\Gamma \setminus \Sigma_2$ close to Σ_2. Then p satisfies (2.2) and (2.5) for any section of Lagrangean spaces in $Tp_k^{-1}(0)$ at $\Gamma \setminus \Sigma_2$ by the invariance, since $\nabla p_k \ne 0$ and $\partial^\alpha p_k = \mathcal{O}(1)$, $\forall\, \alpha$. A bicharacteristic $\Gamma \subset \Sigma \setminus \Sigma_2$ of p is a bicharacteristic for p_k for some k. Then if p_s is the subprincipal symbol and (2.9) is satisfied for a sequence of bicharacteristics of p_k converging in C^∞ to Σ_2, we obtain that the operator is not solvable at any limit of these bicharacteristics at Σ_2.

Example 3.2. Assume that $p(x, \xi)$ is real and vanishes of exactly order $k \ge 2$ at the involutive submanifold $\Sigma_2 = \{\, \xi' = 0 \,\}$, $\xi = (\xi', \xi'') \in \mathbf{R}^m \times \mathbf{R}^{n-m}$, such that the localization

$$\eta \mapsto \sum_{|\alpha| = k} \partial_{\xi'}^\alpha p(x, 0, \xi'') \eta^\alpha$$

is of principal type when $\eta \ne 0$. Then the bicharacteristics of p satisfies (2.2) and (2.5) with $L_j = \{\, \xi = 0 \,\}$ at any point. In fact, $|\partial_{\xi'} p(x, \xi)| \cong |\xi'|^{k-1}$ and $\partial_{x, \xi''} p(x, \xi) = \mathcal{O}(|\xi'|^k)$ so we obtain this since $H_{\widetilde{p}} = \partial_{\xi'} \widetilde{p} \partial_{x'} + \mathcal{O}(|\xi'|)$ and $\partial_x^\alpha \nabla p = \mathcal{O}(|\xi'|^{k-1})$ when $|\xi'| \ll 1$ and $|\xi| \cong 1$. The operator is not solvable if the imaginary part of the subprincipal symbol $\operatorname{Im} p_s$ changes sign from $-$ to $+$ along a convergent sequence of bicharacteristics of the principal symbol and vanishes of at most order $k - 2$ at Σ_2. In particular we obtain the results of [5] and [6].

Example 3.3. As in Example 2.3, let P have real principal symbol $p = w_1^k - a(w')$ in the coordinates $(w_1, w') \in T^*\mathbf{R}^n \setminus 0$, where $w_1 = a(w') = 0$ at Σ_2. We find that (2.5) is satisfied if $dw_1\big|_{L_j} = 0$ and $d\nabla a\big|_{L_j} = \mathcal{O}(|\nabla a|)$ when $w_1 = 0$.

Example 3.4. Let Q be a real and hyperbolic quadratic form on $T^*\mathbf{R}^n$. Then by [2, Theorem 1.4.6] we have the following normal form $Q_1(x,\xi)+Q_2(y,\eta)$ where the symplectic coordinates $(x, y; \xi, \eta) \in T^*(\mathbf{R}^m \times \mathbf{R}^r)$,

$$Q_1(x,\xi) = \sum_{j=1}^{k} \mu_j(x_j^2 + \xi_j^2) + \sum_{j=k+1}^{m} \xi_j^2 \qquad \mu_j > 0 \quad \forall\, j$$

is positive semidefinite and $Q_2(y, \eta)$ is either $-\eta_1^2$, $\mu_0 y_1 \eta_1$ with $\mu_0 \neq 0$ or $2\eta_1\eta_2 - y_2^2$. To simplify the notation, we will assume $\mu_j = 1$, $\forall\, j$.

Now the Hamilton vector field $H_Q(x, y; \xi, \eta) = H_{Q_1}(x, \xi) + H_{Q_2}(y, \eta)$ so the flow of H_Q is a direct sum of the flows of H_{Q_1} and H_{Q_2}. For H_{Q_1} it is a direct sum of the flows on the circles in (x_j, ξ_j) of radius $r_j \to 0$, $j \le k$, and the flows on x_j lines, $k < j \le m$. The circles can only converge in C^∞ to the origin if the radii r_j go to zero fast enough, see Example 2.2. The possible limits are then points or lines in the $x_{k+1}, \dots, x_m$ space.

In the case $Q_2(y, \eta) = -\eta_1^2$ we find that the limit bicharacteristics are given by

$$y_1 = \lambda t,\ x_j = \lambda_j t + a_j,\ k < j \le m \tag{3.1}$$

where $\lambda^2 = \sum_{j=k+1}^m \lambda_j^2$. We can only find a Lagrangean space satisfying (2.5) if $\mu_j = 0$, $\forall\, j$, since such a space cannot be contained in the subspace $\{\, x_j = \xi_j = 0 \,\}$. Then the grazing Lagrangean space can be chosen as $\{\, (x, y; 0, 0) \,\}$ at every point of the bicharacteristic. Theorem 2.9 then gives that the operator with homogeneous principal symbol equal to $Q(x, y; \xi, \eta)$ is not solvable when the imaginary part of the subprincipal symbol changes sign on the lines on Σ_2 given by (3.1), which also follows from the results in [6].

If $Q_2(y, \eta) = y_1\eta_1$, then $Q^{-1}(0) = \{\, y_1\eta_1 = -Q_1(x, \xi) = -\lambda^2 \,\}$ where $|H_{Q_1}| \cong \lambda$. The Hamilton vector field of Q_2 is given by $H_{Q_2} = y_1\partial_{y_1} - \eta_1\partial_{\eta_1}$ which has flow $t \mapsto (y_1 e^t; \eta_1 e^{-t})$ with starting point (y_1, η_1). Here $y_1\eta_1 = -\lambda^2 < 0$ which gives $|y_1| + |\eta_1| \gtrsim |y_1\eta_1|^{1/2} \gtrsim \lambda$. We find that $|H_Q| \cong L = |y_1| + |\eta_1|$, so in order for the bicharacteristic to converge in C^∞, we find from (2.2) that $(|y_1| + |\eta_1|)/L^k = L^{1-k} \lesssim 1$, $\forall\, k$. But then $|y_1| + |\eta_1| \not\to 0$ so the bicharacteristics do not converge in C^∞ to a limit bicharacteristic at Σ_2. Observe that a hyperbolic operator with Hessian of the principal symbol equal to $Q(x, y; \xi, \eta)$ at $\Sigma_2 \bigcap S^*\mathbf{R}^n$ is *effectively hyperbolic* and is solvable with any lower order terms, see the Notes to Chapter 23 in [4].

Finally, when $Q_2(y, \eta) = 2\eta_1\eta_2 - y_2^2$, then the characteristics in the (y, η) variables are $\{\, \eta_1\eta_2 = (y_2^2 + \lambda^2)/2 \,\}$ where $\lambda^2 = Q_1$. We find $H_{Q_2} = 2\eta_2\partial_{y_1} + 2\eta_1\partial_{y_2} + 2y_2\partial_{\eta_2}$ so η_1 is constant on the orbits. Note that $y_2^2 + \lambda^2 = 2\eta_1\eta_2 \le \eta_1^2 + \eta_2^2$. Thus when $|\eta_1| \gtrsim |\eta_2|$ we find that $|\eta_1| \gtrsim |y_2| + \lambda$ on Σ. Now $|H_{\widetilde{Q}_2}\nabla\widetilde{Q}_2| \gtrsim \eta_1^{-1}$ so in order for (2.2) to hold we must have $|\eta_1| \gtrsim 1$, so the characteristics will in that case not converge in C^∞ to

any limit bicharacteristic at Σ_2. When $|\eta_2| \gg |\eta_1|$ we find that $|\eta_2| \gg |y_2| + \lambda$ on Σ. A straighforward computation shows that applying $H_{\widetilde{Q}}$ twice on $\nabla \widetilde{Q}$ gives a factor η_1/η_2^3. Thus, by (2.2) we find that the bicharacteristics converge in C^∞ to the y_1 lines on Σ_2 only if $|\eta_1| \lesssim |\eta_2|^3$, which implies that $|y_2| + \lambda \lesssim \eta_2^2$. Then $H_{\widetilde{Q}} = \partial_{y_1} + \mathcal{O}(|\eta_2|)$ so the bicharacteristics can only converge to y_1 lines at $\eta_2 \to 0$. An example is the case when $y_2 = \eta_1 = \lambda = 0$ and $\eta_2 \to 0$, then $H_{Q_2} = 2\eta_2\partial_{y_1}$ and the grazing Lagrangean space can be chosen as $\{\, (x, y_1, 0; 0, 0, \eta_2) \,\}$ at every point of the bicharacteristics. Thus Theorem 2.9 gives that the operator with homogeneous principal symbol equal to $Q(x, y; \xi, \eta)$ when $|\xi|^2 + |\eta|^2 = 1$ is not solvable when the imaginary part of the subprincipal symbol changes sign on y_1 lines at Σ_2.

4. The normal form

First we shall put the adjoint operator P^* on a normal form uniformly and microlocally near the bicharacteristics Γ_j converging to Γ. This will present some difficulties since we only have conditions at the bicharacteristics. By the invariance, we may multiply with an elliptic operator so that the order of P^* is $m = 1$ and P^* has the symbol expansion $p + p_0 + \dots$, where p is the principal symbol. As in Remark 2.7 we may assume that p_0 is the subprincipal symbol. Observe that for the adjoint the signs in (2.9) are reversed and it changes to

$$\max_{\partial\Gamma_j} \int \operatorname{Im} p_0 |H_p|^{-1}\, ds / |\log \kappa_j| \to -\infty \qquad j \to \infty \tag{4.1}$$

where κ_j given by (2.8). By changing w_j to the maximum of the integral in (4.1) only improves the estimate so we may assume that

$$\int \operatorname{Im} p_0/|H_p|\, ds \le 0 \qquad \text{on } \Gamma_j \tag{4.2}$$

with equality at $w_j \in \Gamma_j$. Since $\nabla \operatorname{Im} p_0$ and ∇H_p are bounded on S^*X and $|H_p| \ge \kappa_j$ on Γ_j, we find that $|H_p|$ and $\operatorname{Im} p_0/|H_p|$ only change with a fixed factor and a bounded term on an interval of length $\ll \kappa_j$ on Γ_j. Therefore, we find that integrating $\operatorname{Im} p_0/|H_p|$ over such intervals only gives bounded terms. Thus, we find from (2.10) that

$$|\Gamma_j| \gg \kappa_j \tag{4.3}$$

and that condition (2.9) holds on intervals of length $\cong \kappa_j$ at the endpoints of Γ_j.

Now we choose

$$1 \lesssim \lambda_j = \kappa_j^{-1/\varepsilon} \Leftrightarrow \kappa_j = \lambda_j^{-\varepsilon} \tag{4.4}$$

for some $\varepsilon > 0$ to be determined later. Then we may replace$|\log \kappa_j|$ with $\log \lambda_j$ in (2.9)–(2.10). By choosing a subsequence and renumbering, we may assume by (2.9) that

$$\min_{\partial\Gamma_j} \int \operatorname{Im} p_0/|H_p|\, ds \le -j \log \lambda_j \tag{4.5}$$

and that this holds on intervals of length $\cong \kappa_j$ at the endpoints of Γ_j. Next, we introduce the normalized principal and subprincipal symbols

$$\widetilde{p} = p/|H_p| \qquad \text{and} \qquad p_s = p_0/|H_p|$$

then we have that $H_{\widetilde{p}}\big|_{\Gamma_j} \in C^\infty$ uniformly, $|H_{\widetilde{p}}| = 1$ on Γ_j and $dH_{\widetilde{p}}\big|_{L_j}$ is uniformly bounded at Γ_j. We find that condition (4.5) becomes

$$\min_{\partial\Gamma_j} \int \operatorname{Im} p_s \, ds \le -j \log \lambda_j \tag{4.6}$$

Observe that because of condition (2.10) we have that $\partial\Gamma_j$ has two components since $\operatorname{Im} p_s$ has opposite sign there, so Γ_j is a uniformly embedded curve.

In the following we shall consider a fixed bicharacteristic Γ_j and suppress the index j, so that $\Gamma = \Gamma_j$, $L = L_j$ and $\kappa = \kappa_j = \lambda^{-\varepsilon}$. Observe that the preparation will be uniform in j. Now $H_{\widetilde{p}} \in C^\infty$ uniformly on Γ but not in a neighborhood. By (2.5) we may define the first jet of $\widetilde{p}$ at Γ uniformly. Since $\Gamma \in C^\infty$ uniformly, we can choose local coordinates uniformly so that $\Gamma = \{\, (t,0)) : \ t \in I \subset \mathbf{R} \,\}$ locally. In fact, take a local parametrization $\gamma(t)$ of Γ with respect to the arclength and choose the orthogonal space M to the tangent vector of Γ at a point w_0 with respect to some local Riemannean metric. Then $\mathbf{R} \times M \ni (t, w) \mapsto \gamma(t) + w$ is uniformly bounded in C^∞ with uniformly bounded inverse near $(t_0, 0)$ giving local coordinates near Γ. We may then complete t to a symplectic coordinate system uniformly. We can define the first order Taylor term of $\widetilde{p}$ at Γ by

$$\varrho(t,w) = \partial_w \widetilde{p}(t,0) \cdot w \qquad w = (x, \tau, \xi) \tag{4.7}$$

which is uniformly bounded. This can be done locally, and by using a uniformly bounded partition of unity we obtain this in a fixed neighborhood of Γ. Going back to the original coordinates, we find that $\varrho \in C^\infty$ uniformly near Γ and $\widetilde{p} - \varrho = \mathcal{O}(d^2)$, but the error is not uniformly bounded. Here d is the homogeneous distance to Γ, i.e., the distance with respect to the homogeneous metric $dt^2 + |dx|^2 + (d\tau^2 + |d\xi|^2)/\langle(\tau,\xi)\rangle^2$. But by condition (2.5) we find that the second order derivatives of $\widetilde{p}$ along the Lagrangean space L at Γ are uniformly bounded. We shall use homogeneous coordinates, i.e., local coordinates which are normalized with respect to the homogeneous metric.

By completing $\tau = \varrho$ in (4.7) to a uniformly bounded homogeneous symplectic coordinate system near Γ and conjugating with the corresponding uniformly bounded Fourier integral operator we may assume that

$$\Gamma = \{\, (t, 0; 0, \xi_0) : \ t \in I \,\} \tag{4.8}$$

for $|\xi_0| = 1$, some fixed interval $I \ni 0$, and $\widetilde{p} \cong \tau$ modulo second order terms at Γ. The second order terms are not uniformly bounded, but $d\nabla\widetilde{p}\big|_L$ is uniformly bounded at Γ by (2.5). Since $d\widetilde{p} = d\tau$ on Γ we find that $H_{\widetilde{p}}\big|_\Gamma = D_t$ and we may obtain that $L = \{\, (t, x; 0, 0) \,\}$ at any given point at Γ by choosing linear symplectic coordinates (x, ξ).

Let

$$q(t,w) = |\nabla p(t,w)| \ge \kappa = \lambda^{-\varepsilon} \qquad \text{at } \Gamma \tag{4.9}$$

and extended so that q is homogeneous of degree 0, which is the norm of the homogeneous gradient of p. Recall that λ is a parameter that depends on the bicharacteristic Γ. We shall change coordinates so that $(t,w) = 0$ corresponds to the point $(0,0;0,\xi_0) \in \Gamma$ given by (4.8), and then localize in neighborhoods depending on λ. We have $|\nabla \widetilde{p}| \equiv 1$ at Γ, higher derivatives are not uniformly bounded but can be handled by the using the metric

$$g_\varepsilon = (dt^2 + |dw|^2)\lambda^{2\varepsilon}$$

and the symbol classes $f \in S(m, g_\varepsilon)$ defined by $\partial^\alpha f = \mathcal{O}(m\lambda^{|\alpha|\varepsilon})$, $\forall\, \alpha$.

Proposition 4.1. *We have that q is a weight for g_ε, $q \in S(q, g_\varepsilon)$ and $\widetilde{p}(t,w) \in S(\lambda^{1-\varepsilon}, g_\varepsilon)$ when $|w| \le c\lambda^{-\varepsilon}$ for some $c > 0$ when $t \in I$.*

This gives $p = q\widetilde{p} \in S(q\lambda^{1-\varepsilon}, g_\varepsilon)$ when $|w| \le c\lambda^{-\varepsilon}$. Observe that $b \in S^\mu_{1-\varepsilon,\varepsilon}$ if and only if $b \in S(\lambda^\mu, g_\varepsilon)$ in homogeneous coordinates when $|\xi| \gtrsim \lambda \gtrsim 1$. In fact, in homogeneous coordinates z this means that $\partial_z^\alpha b = \mathcal{O}(|\xi|^{\mu+|\alpha|\varepsilon})$ when $|z| \cong 1$.

Proof. We shall use the previously chosen coordinates (t,w) so that $\Gamma = \{\, (t,0) :\ t \in I \,\}$. Now $\partial^2 p = \mathcal{O}(1)$, $q \ge \lambda^{-\varepsilon}$ at Γ by (4.9) and

$$\partial q = \nabla p \cdot (\partial \nabla p)/q \qquad \text{when } q \ne 0 \tag{4.10}$$

which is uniformly bounded. We find that $q(s,w) \cong q(t,0)$ when $|s-t| + |w| \le c\lambda^{-\varepsilon}$ for small enough $c > 0$, so q is a weight for g_ε there. This gives that $|p(t,w)| \lesssim q(t,w)\lambda^{-\varepsilon}$, $|\nabla p(t,w)| = q(t,w)$ and $|\partial^\alpha p| \lesssim 1 \lesssim q\lambda^\varepsilon$ for $|\alpha| \ge 2$, which gives $p \in S(q\lambda^{-\varepsilon}, g_\varepsilon)$ when $|w| \le c\lambda^{-\varepsilon}$.

We find from (4.10) that $\partial q = \mathcal{O}(q\lambda^\varepsilon)$ when $|w| \le c\lambda^{-\varepsilon}$, since $\nabla p \in S(q, g_\varepsilon)$ in this domain. By induction over the order of differentiation we obtain from (4.10) that $q \in S(q, g_\varepsilon)$ when $|w| \le c\lambda^{-\varepsilon}$, which gives the result. $\square$

Next, we put $Q(t,w) = \lambda^\varepsilon \widetilde{p}(t\lambda^{-\varepsilon}, w\lambda^{-\varepsilon})$ when $t \in I_\varepsilon = \{\, t\lambda^\varepsilon :\ t \in I \,\}$. Then by Proposition 4.1 we find that $Q \in C^\infty$ uniformly when $|w| \lesssim 1$ and $t \in I_\varepsilon$, $\partial_\tau Q \equiv 1$ and $|\partial_{t,x,\xi} Q| \equiv 0$ when $w = 0$ and $t \in I_\varepsilon$. Thus we find $|\partial_\tau Q| \ne 0$ for $|w| \lesssim 1$ and $t \in I_\varepsilon$. By using Taylor's formula at Γ we can write $Q(t,x;\tau,\xi) = \tau + h(t,x;\tau,\xi)$ when $|w| \lesssim 1$ and $t \in I_\varepsilon$, where $h = |\nabla h| = 0$ at $w = 0$. By using the Malgrange preparation theorem, we find

$$\tau = a(t,w)(\tau + h(t,w)) + s(t,x,\xi) \qquad |w| \lesssim 1 \quad t \in I_\varepsilon$$

where a and $s \in C^\infty$ uniformly, $a \ne 0$, and on Γ we have $a = 1$ and $s = |\nabla s| = 0$. In fact, this can be done locally in t and by a uniform partition of unity for $t \in I_\varepsilon$. This gives

$$a(t,w)Q(t,w) = \tau - s(t,x,\xi) \qquad |w| \lesssim 1 \quad t \in I_\varepsilon \tag{4.11}$$

In the original coordinates, we find that

$$\lambda^{\varepsilon}\widetilde{p}(t,w) = a^{-1}(t\lambda^{\varepsilon}, w\lambda^{\varepsilon})(\tau\lambda^{\varepsilon} - s(t\lambda^{\varepsilon}, x\lambda^{\varepsilon}, \xi\lambda^{\varepsilon}))$$

and thus

$$\widetilde{p}(t,w) = b(t,w)(\tau - r(t,x,\xi)) \qquad |w| \lesssim \lambda^{-\varepsilon} \quad t \in I \tag{4.12}$$

where $0 \neq b \in S(1, g_\varepsilon)$, $r(t,x,\xi) = \lambda^{-\varepsilon}s(t\lambda^{\varepsilon}, x\lambda^{\varepsilon}, \xi\lambda^{\varepsilon}) \in S(\lambda^{-\varepsilon}, g_\varepsilon)$ when $|w| \lesssim \lambda^{-\varepsilon}$, and $t \in I$, $b = 1$ and $r = |\nabla r| = 0$ on Γ. By condition (2.5) we also find that

$$\left|d\nabla r\big|_L\right| \le C \qquad \text{when } w = 0 \text{ and } t \in I \tag{4.13}$$

Recall that $\widetilde{p} = p/q$, where $q \in S(q, g_\varepsilon)$ when $|w| \lesssim \lambda^{-\varepsilon}$.

In the following, we shall denote by Γ the rays in T^*X that goes through the bicharacteristic. By homogeneity we obtain from (4.12) that

$$b^{-1}q^{-1}p(t,x;\tau,\xi) = \tau - r(t,x,\xi)$$

where $b^{-1} \in S^0_{1-\varepsilon,\varepsilon}$, $q^{-1} \in S^{\varepsilon}_{1-\varepsilon,\varepsilon}$ and $\tau - r \in S^{1-\varepsilon}_{1-\varepsilon,\varepsilon}$ when $|\xi| \gtrsim \lambda$ and the homogeneous distance d to Γ is less than $c|\xi|^{-\varepsilon}$ for some $c > 0$.

Next we take a cut-off function $\chi \in S^0_{1-\varepsilon,\varepsilon}$ supported where $d \lesssim |\xi|^{-\varepsilon}$ and $|\xi| \gtrsim \lambda$ so that $b \ge c_0 > 0$ in $\operatorname{supp}\chi$ and $\chi = 1$ when $|\xi| \ge c\lambda$ and $d \le c|\xi|^{-\varepsilon}$ for any chosen $c > 0$. We let $B = \chi b^{-1}q^{-1} \in S^{\varepsilon}_{1-\varepsilon,\varepsilon}$ and compose the pseudodifferential operator B with P^*. Since $P^* \in \Psi^1_{1,0}$ we obtain an asymptotic expansion of BP^* in $S^{1+\varepsilon-j(1-\varepsilon)}_{1-\varepsilon,\varepsilon}$ for $j = 0$, 1, 2,… when $d \lesssim |\xi|^{-\varepsilon}$. But actually the symbol is in a better class. The principal symbol is

$$(\tau - r(t,x,\xi))\chi \in S^{1-\varepsilon}_{1-\varepsilon,\varepsilon} \qquad d \lesssim |\xi|^{-\varepsilon}$$

and the calculus gives that the subprincipal symbol is equal to

$$\frac{i}{2}H_p(\chi b^{-1}q^{-1}) + \chi b^{-1}q^{-1}p_0 \tag{4.14}$$

where p_0 is the subprincipal symbol of P^*. As before, we shall use homogeneous coordinates when $|\xi| \gtrsim \lambda$. Then Proposition 4.1 gives $p = q\widetilde{p} \in S(q\lambda^{1-\varepsilon}, g_\varepsilon)$ and since $\chi b^{-1}q^{-1} \in S(q^{-1}, g_\varepsilon)$ we find that (4.14) is in $S(\lambda^{\varepsilon}, g_\varepsilon)$ when $d \lesssim \lambda^{-\varepsilon}$ and $|\xi| \gtrsim \lambda$. The value of H_p at Γ is equal to $q\partial_t$ so the value of (4.14) is equal to

$$\frac{1}{2i}\partial_t q/q + p_0/q = \frac{D_t|\nabla_h p|}{2|\nabla_h p|} + \frac{p_0}{|\nabla_h p|} \qquad \text{when } |\xi| \gtrsim \lambda \text{ at } \Gamma \tag{4.15}$$

where $|\nabla_h p| = \sqrt{|\partial_x p|^2/|\xi|^2 + |\partial_\xi p|^2}$ is the homogeneous gradient, and the error of this approximation is bounded by $\lambda^{2\varepsilon}$ times the homogeneous distance d to Γ. In fact, since $\partial p \in S(q, g_\varepsilon)$ by Proposition 4.1, we have $H_p q^{-1} \in S(\lambda^{\varepsilon}, g_\varepsilon)$ and $p_0/q \in S(q^{-1}, g_\varepsilon)$. Observe that $p_0/|\nabla_h p|$ is equal to the normalized subprincipal symbol of P^* on S^*X. This preparation can only be done when $|\xi| \gtrsim \lambda$ and the homogeneous distance to Γ is less than $c|\xi|^{-\varepsilon}$, but we have to estimate the error terms.

Definition 4.2. For $\varepsilon < 1/2$ and $R \in \Psi^{\mu}_{\varrho,\delta}$, where $\varrho > \varepsilon$ and $\delta < 1-\varepsilon$, we say that $T^*X \ni (x_0,\xi_0) \notin \mathrm{WF}_\varepsilon(R)$ if the symbol of R is $\mathcal{O}(|\xi|^{-N})$, $\forall\, N$, when the homogeneous distance to the ray $\{\, (x_0, \varrho\xi_0) : \ \varrho \in \mathbf{R}_+ \,\}$ is less than $c|\xi|^{-\varepsilon}$ for some $c > 0$.

By the calculus, this means that there exists $A \in \Psi^0_{1-\varepsilon,\varepsilon}$ so that $A \ge c > 0$ in a neighborhood of the ray such that $AR \in \Psi^{-N}$ for any N. This neighborhood is in fact the points with fixed bounded homogeneous distance with respect to the metric g_ε when $\lambda \cong |\xi|$. It also follows from the calculus that this definition is invariant under composition with classical pseudodifferential operators and conjugation with elliptic homogeneous Fourier integral operators since the conjugated symbol is given by an asymptotic expansion by the conditions on ϱ and δ. We also have that $\mathrm{WF}_\varepsilon(R) \subset \mathrm{WF}(R)$ when $R \in \Psi^{\mu}_{\varrho,\delta}$.

Now we can use the Malgrange division theorem in order to make the lower order terms independent on τ when $d \lesssim |\xi|^{-\varepsilon} \lesssim \lambda^{-\varepsilon}$, starting with the subprincipal symbol $\widetilde{p}_0 \in S^{\varepsilon}_{1-\varepsilon,\varepsilon}$ of BP^* given by (4.14). Then rescaling as before so that $Q_0(t,w) = \lambda^{-\varepsilon}\widetilde{p}_0(t\lambda^{-\varepsilon}, w\lambda^{-\varepsilon})$ we obtain that

$$Q_0(t,w) = \widetilde{c}(t,w)(\tau - s(t,x,\xi)) + \widetilde{q}_0(t,x,\xi) \qquad |w| \lesssim 1 \quad t \in I_\varepsilon$$

where s is given by (4.11), and $\widetilde{c}$ and $\widetilde{q}_0$ are uniformly in C^∞. This can be done locally and by a partition of unity for $t \in I_\varepsilon$. We find in the original coordinates that

$$\widetilde{p}_0(t,w) = c(t,w)(\tau - r(t,x,\xi)) + q_0(t,x,\xi) \qquad d \lesssim \lambda^{-\varepsilon} \quad t \in I \tag{4.16}$$

where $q_0(t,w) = \lambda^{\varepsilon}\widetilde{q}_0(t\lambda^{\varepsilon}, w\lambda^{\varepsilon}) \in S(\lambda^{\varepsilon}, g_\varepsilon)$ and $c(t,w) = \lambda^{2\varepsilon}\widetilde{c}(t\lambda^{\varepsilon}, w\lambda^{\varepsilon}) \in S(\lambda^{2\varepsilon}, g_\varepsilon)$. By using a uniform partition of unity in $\Psi^0_{1-\varepsilon,\varepsilon}$, we obtain (4.16) uniformly when $|\xi| \gtrsim \lambda$ and the homogeneous distance to Γ is $\lesssim |\xi|^{-\varepsilon}$ with $c \in S^{2\varepsilon-1}_{1-\varepsilon,\varepsilon}$ by homogeneity, $q_0 \in S^{\varepsilon}_{1-\varepsilon,\varepsilon}$ and $q_0 = \widetilde{p}_0$ at Γ. Now the composition of the operators having symbols c and $\tau - r$ gives error terms that are in $S^{3\varepsilon-1}_{1-\varepsilon,\varepsilon}$ when $|\xi| \gtrsim \lambda$. Thus if $\varepsilon < 1/3$ then by multiplication with an pseudodifferential operator with symbol $1-c$ we can make the subprincipal symbol independent of τ. By iterating this procedure we can successively make any lower order terms independent of τ when $|\xi| \gtrsim \lambda$ and the homogeneous distance d to Γ is less than $c|\xi|^{-\varepsilon}$. By applying the cut-off function χ we obtain the following result.

Proposition 4.3. *By conjugating with an elliptic homogeneous Fourier integral operator we may obtain that Γ is given by (4.8). If $0 < \varepsilon < 1/3$ then for any $c > 0$ we may multiply with an homogeneous elliptic operator $B \in \Psi^{\varepsilon}_{1-\varepsilon,\varepsilon}$ to obtain that $BP^* = Q + R \in \Psi^{1-\varepsilon}_{1-\varepsilon,\varepsilon}$ where $\Gamma \bigcap \mathrm{WF}_\varepsilon(R) = \emptyset$ and the symbol of Q is equal to*

$$\tau - r(t,x,\xi) + q_0(t,x,\xi) + r_0(t,x,\xi) \qquad \textit{when } d \le c|\xi|^{-\varepsilon},\ |\xi| \ge c\lambda \textit{ and } t \in I \tag{4.17}$$

Here $r \in S^{1-\varepsilon}_{1-\varepsilon,\varepsilon}$, $q_0 \in S^{\varepsilon}_{1-\varepsilon,\varepsilon}$ and $r_0 \in S^{3\varepsilon-1}_{1-\varepsilon,\varepsilon}$, $r = |\nabla r| = 0$ on Γ, and q_0 is equal to

$$\frac{D_t|\nabla_h p(t,0)|}{2|\nabla_h p(t,0)|} + \frac{p_0(t,0)}{|\nabla_h p(t,0)|} + \mathcal{O}(\lambda^{2\varepsilon} d) \qquad \textit{when } d \le c|\xi|^{-\varepsilon},\ |\xi| \ge c\lambda \textit{ and } t \in I \tag{4.18}$$

where d is the homogeneous distance to Γ and $|\nabla_h p| = \sqrt{|\partial_x p|^2/|\xi|^2 + |\partial_\xi p|^2}$ is the homogeneous gradient. This preparation is uniform with respect to $\lambda \ge 1$.

Observe that the integration of the term $D_t|\nabla_h p(t,0)|/2|\nabla_h p(t,0)|$ in (4.18) will give terms that are $\mathcal{O}(\log(|\nabla_h p(t,0)|)) = \mathcal{O}(|\log(\lambda)|+1)$, which do not affect condition (2.9). We find from Proposition 4.3 that $\widetilde{p}(t,x;\tau,\xi) \cong \tau - r(t,x,\xi)$ modulo terms vanishing of third order at Γ since r vanishes of second order at Γ.

Recall that L is a smooth section of Lagrangean spaces $L(w) \subset T_w\Sigma \subset T_w(T^*\mathbf{R}^n)$, $w \in \Gamma$, such that the linearization of the Hamilton vector field H_p is in TL at Γ. By Proposition 4.3 we may assume that $\Gamma = \{\, (t,0;0,\xi_0) : \ t \in I \,\}$, $0 \in I$, and $\widetilde{p}(t,x;\tau,\xi) = \tau - r(t,x,\xi)$ modulo terms vanishing of third order at Γ. Then we may parametrize $L(t) = L(w)$ where $w = (t,0,\xi_0)$ for $t \in I$. Now since $T^*\mathbf{R}^n$ is a linear space, we may identify the fiber of $T_w(T^*\mathbf{R}^n)$ with $T^*\mathbf{R}^n$. Since $L(w) \subset T_w\Sigma$ and $w \in \Gamma$ we find that $\tau = 0$ in $L(w)$. Since $L(w)$ is Lagrangean, we find that t lines are in $L(w)$. By choosing symplectic coordinates in (x,ξ) we obtain that $L(0) = \{\, (s,y;0,0) : \ (s,y) \in \mathbf{R}^n \,\}$, then by condition (2.5) we find that $\partial_x^2 r(0,0,\xi_0)$ is uniformly bounded. Since $\tau = 0$ on $L(t)$ and $L(t)$ is Lagrangean we find for small t by continuity that

$$L(t) = \{\, (s,y;0,A(t)y) : \ (s,y) \in \mathbf{R}^n \,\} \tag{4.19}$$

where $A(t)$ is real, continuous and symmetric for $t \in I$ and $A(0) = 0$. Since the linearization of the Hamilton vector field H_p at Γ is tangent to L, we find that L is parallel under the flow of that linearization. Since $L(t)$ is Lagrangean it is only the restriction of the second order jet of $r(t,w)$ to $L(t)$ that determines the evolution of $t \mapsto L(t)$. For (4.19) this restriction is given by the second order Taylor expansion of

$$R(t,x) = r(t,x,\xi_0 + A(t)x)$$

thus $\partial_x^2 R(t,0)$ is uniformly bounded by condition (2.5). The linearized Hamilton vector field is

$$\begin{aligned}\partial_t &+ \langle \partial_x^2 R(t,0)x, \partial_\xi \rangle \\ &= \partial_t + \langle \big(\partial_x^2 r(t,0,\xi_0) + 2\,\mathrm{Re}(\partial_x\partial_\xi r(t,0,\xi_0)A) + A\partial_\xi^2 r(t,0,\xi_0)A\big)\, x, \partial_\xi \rangle\end{aligned}$$

where $\mathrm{Re}\, B = (B + B^t)/2$ is the symmetric part of B. Applying this on $\xi - A(t)x$, which vanishes identically on $L(t)$ for $t \in I$, we obtain that

$$-A'(t) + \partial_x^2 r(t,0,\xi_0) + 2\,\mathrm{Re}(\partial_x\partial_\xi r(t,0,\xi_0)A(t)) + A(t)\partial_\xi^2 r(t,0,\xi_0)A(t) = 0$$

which gives the evolution of $L(t)$. The equation

$$A'(t) = \partial_x^2 r(t,0,\xi_0) + 2\,\mathrm{Re}(\partial_x\partial_\xi r(t,0,\xi_0)A(t)) + A(t)\partial_\xi^2 r(t,0,\xi_0)A(t) \qquad A(0) = 0 \tag{4.20}$$

is locally uniquely solvable and the right-hand side is uniformly bounded as long as A is bounded. Observe that by uniqueness, $A(t) \equiv 0$ if and only if $\partial_x^2 r(t,0,\xi_0) \equiv 0$, $\forall\, t$ (see also Example 2.6). But since (4.20) is non-linear, the solution could become

unbounded if $\partial_x^2 r \neq 0$ and $\partial_\xi^2 r \neq 0$ so that $\|A(s)\| \to \infty$ as $s \to t_1 \in I$. This means that the angle between $L(t) = \{\, (s, y; 0, A(t)y) : \ (s, y) \in \mathbf{R}^n \,\}$ and the vertical space $\{\, (s, 0; 0, \eta) : (s, \eta) \in \mathbf{R}^n \,\}$ goes to zero, but this is just a coordinate singularity.

In general, since we identify the fiber of $T_w(T^*\mathbf{R}^n)$ with $T^*\mathbf{R}^n$ we may define $R(t, x, \xi)$ for each t so that

$$R(t, x, \xi) = r(t, x, \xi_0 + \xi) \quad \text{when } (0, x; 0, \xi) \in L(t) \tag{4.21}$$

Then $R = r$ on L and we find that

$$\tau - \langle R(t)z, z\rangle/2 \in C^\infty \tag{4.22}$$

if $z = (x, \xi)$ and $R(t) = \partial_z^2 R(t, 0, 0)\big|_L(t)$. Observe that we find from (2.5) that (4.22) is uniformly in C^∞ in z. We find that $R(0) = \partial_x^2 r(t, 0, \xi_0)$ and in general $R(t)$ is given by the right hand side of (4.20). Now we can complete t and (4.22) to a uniform homogeneous symplectic coordinates system so that $\Gamma = \{\, (t, 0, \xi_0) : \ t \in I \,\}$ and $L(0) = \{\, (s, y; 0, 0) : \ (s, y) \in \mathbf{R}^n \,\}$. In fact, we may let x and ξ have the same values when $t = 0$ and clearly H_τ is is not changed on Γ since then $z = 0$. This is a uniformly bounded linear symplectic transformation in (x, ξ) which is uniformly C^1 in t. It is given by a uniformly bounded elliptic Fourier integral operator $F(t)$ which has uniformly bounded t derivative. We will call this type of Fourier integral operator a C^1 *section of Fourier integral operators*. This will give uniformly bounded terms when we conjugate a first order differential operator in t with $F(t)$, for example the normal form which has symbol given by (4.17). For t close to 0 the section $F(t)$ is given by multiplication with $e^{\langle A(t)x,x\rangle/2}$, where $A(t)$ solves (4.20). For general t we can put $F(t)$ on this form after a linear symplectic transformation in (x, ξ). Observe that $F(t)$ is continuous on local L^2 Sobolev spaces in x, uniformly in t, since it is continuous with respect to the norm $\|(1 + |x|^2 + |D|^2)^k u\|$, $\forall\, k$. In fact, it suffices to check this for the generators of the group of Fourier integral operators corresponding to linear symplectic transformations, which are given by the partial Fourier transforms, linear transformations in x and multiplication with $e^{i\langle Ax,x\rangle}$ where A is real and symmetric.

We find in the new coordinates that $p = \tau - r_1$, where $r_1(t, x, \xi)$ is independent of τ and satisfies $\partial_z^2 r_1(t, 0, 0)\big|_L(t) \equiv 0$. This follows since

$$p(t, x; \tau, \xi) = \tau - \langle R(t)z, z\rangle/2 - r_1(t, x, \tau, \xi)$$

where $\partial_\tau r_1 = -\{\, t, r_1 \,\} = -\{\, t, r \,\} \equiv 0$ is invariant under changes of symplectic coordinates. Similarly we find that the lower order terms $p_j(t, x, \xi)$ are independent of τ for $j \leq 0$. Since the evolution of L is determined by the second order derivatives of the principal symbol along L by Example 2.6, we find that $L(t) \equiv \{\, (t, x; 0, 0) : \ (t, x) \in \mathbf{R}^n \,\}$. Changing notation so that $r = r_1$ and $p(t, x; \tau, \xi) = \tau - r(t, x, \xi)$ we obtain the following result.

Proposition 4.4. *By conjugation with a uniformly bounded C^1 section of elliptic Fourier integral operators, corresponding to linear symplectic transformations in (x,ξ), we may assume that the symplectic coordinates are chosen so that the grazing Lagrangean space $L(w) \equiv \{\, (t,x,0,0): \ (t,x) \in \mathbf{R}^n \,\}$, $\forall\, w \in \Gamma$, which implies that $\partial_x^2 r(t,0) \equiv 0$.*

We shall apply the adjoint P^* of the operator on the form in Proposition 4.3 on approximate solutions on the form

$$u_\lambda(t,x) = \exp(i\lambda(\langle x,\xi_0\rangle + \omega(t,x))) \sum_{j=0}^{M} \varphi_j(t,x)\lambda^{-j\varrho} \tag{4.23}$$

where $|\xi_0| = 1$ and the phase function $\omega(t,\cdot) \in S(\lambda^{-7\varepsilon}, g_{3\varepsilon})$ is real valued such that $\partial_x^2\omega(t,0) \equiv 0$ and $\varphi_j(t,x) \in S(1,g_\delta)$ has support where $|x| \lesssim \lambda^{-\delta}$. Here δ, ε and ϱ are positive constants to be determined later. The phase function $\omega(t,x)$ will be constructed in Section 5, see Proposition 5.1. Observe that we have assumed that $\varepsilon < 1/3$ in Proposition 4.3, but we shall impose further restrictions on ε later on. We shall assume that $\varepsilon + \delta < 1$, then if $p(t,x,\xi) \in \Psi^{1-\varepsilon}_{1-\varepsilon,\varepsilon}$ we obtain the formal expansion (see [7, Chapter VI, Theorem 3.1])

$$\begin{aligned} &p(t,x,D_x)(\exp(i\lambda(\langle x,\xi_0\rangle + \omega(t,x)))\varphi(t,x)) \\ &\quad \sim \exp(i\lambda(\langle x,\xi_0\rangle + \omega(t,x))) \sum_\alpha \partial_\xi^\alpha p(t,x,\lambda(\xi_0 + \partial_x\omega(t,x)))\mathcal{R}_\alpha(\omega,\lambda,D)\varphi(t,x)/\alpha! \end{aligned} \tag{4.24}$$

where $\mathcal{R}_\alpha(\omega,\lambda,D)\varphi(t,x) = D_y^\alpha(\exp(i\lambda\widetilde{\omega}(t,x,y))\varphi(t,y))\big|_{y=x}$ with

$$\widetilde{\omega}(t,x,y) = \omega(t,y) - \omega(t,x) + (x-y)\partial_x\omega(t,x)$$

Observe that if $|\partial_x\omega| \ll 1$ then this only involves the values of $p(t,x,\xi)$ where $|\xi| \geq c\lambda$ for some $c > 0$. Using this expansion we find that

$$\begin{aligned} &P^*(t,x,D)(\exp(i\lambda(\langle x,\xi_0\rangle + \omega(t,x)))\varphi(t,x)) \\ &\quad \sim \exp(i\lambda(\langle x,\xi_0\rangle + \omega(t,x)))\big(\lambda\partial_t\omega(t,x) - r(t,x,\lambda(\xi_0 + \partial_x\omega))\varphi(t,x) \\ &\quad + D_t\varphi(t,x) - \sum_j \partial_{\xi_j} r(t,x,\lambda(\xi_0+\partial_x\omega))D_{x_j}\varphi(t,x) + q_0(t,x,\lambda(\xi_0+\partial_x\omega))\varphi(t,x) \\ &\quad + \sum_{jk} \partial_{\xi_j}\partial_{\xi_k} r(t,x,\lambda(\xi_0+\partial_x\omega))(D_{x_j}D_{x_k}\varphi(t,x) + i\lambda\varphi(t,x)D_{x_j}D_{x_k}\omega(t,x))/2 + \dots\big) \end{aligned} \tag{4.25}$$

which gives an expansion in $S(\lambda^{1-\varepsilon-j(1-\delta-\varepsilon)}, g_\delta)$, $j \geq 0$, if $\delta + \varepsilon < 1$ and $\varepsilon \leq 1/4$. In fact, since $|\xi| \cong \lambda$ every ξ derivative on terms in $S^{1-\varepsilon}_{1-\varepsilon,\varepsilon}$ gives a factor that is $\mathcal{O}(\lambda^{\varepsilon-1})$ and every x derivative of φ gives a factor that is $\mathcal{O}(\lambda^\delta)$. A factor $\lambda D_x^\alpha\omega$ requires $|\alpha|$ number of ξ derivatives of a term in the expansion of P^*, which gives a factor that is $\mathcal{O}(\lambda^{(2-|\alpha|)(1-4\varepsilon)})$. Similarly, the expansion coming from terms in P^* that have symbols in $S^\varepsilon_{1-\varepsilon,\varepsilon}$ gives an expansion in $S^{\varepsilon-j(1-\delta-\varepsilon)}_{1-\varepsilon,\varepsilon}$, $j \geq 0$. Thus, if $\delta + \varepsilon < 2/3$ and $\varepsilon \leq 1/4$ then the terms in the expansion have negative powers of λ except the terms in (4.25), and for the last ones we

find that

$$\sum_{jk} \partial_{\xi_j}\partial_{\xi_k} r(t,x,\lambda(\xi_0+\partial_x\omega))(D_{x_j}D_{x_k}\varphi + i\lambda\varphi D_{x_j}D_{x_k}\omega) = \mathcal{O}(\lambda^{2\delta+\varepsilon-1}+\lambda^{3\varepsilon-\delta}) \tag{4.26}$$

In fact, $\partial_{\xi_j}\partial_{\xi_k} r(t,x,\lambda(\xi_0+\partial_x\omega)) = \mathcal{O}(\lambda^{\varepsilon-1})$ and $D_{x_j}D_{x_k}\omega = \mathcal{O}(\lambda^{2\varepsilon}d)$ when $\varphi \neq 0$, since we have $D_{x_j}D_{x_k}\omega = 0$ when $x = 0$, and $d = \mathcal{O}(\lambda^{-\delta})$ in $\operatorname{supp}\varphi$.

The error terms in (4.26) are of equal size if $2\delta+\varepsilon-1 = 3\varepsilon-\delta$, i.e., $\delta = (1+2\varepsilon)/3$. We then obtain $3\varepsilon-\delta = (7\varepsilon-1)/3 < 0$ if $\varepsilon < 1/7$. Observe that in this case $1-\delta-\varepsilon = (2-5\varepsilon)/3$ and $\delta+\varepsilon < 2/3$ since $\varepsilon < 1/5$. We also have that $1-4\varepsilon > 1-\delta-\varepsilon$ since $\delta > 3\varepsilon$. Thus we obtain the following result.

Proposition 4.5. *Assume that $\omega(t,\cdot) \in S(\lambda^{-7\varepsilon}, g_{3\varepsilon})$ is real valued and $\partial_x^2\omega(t,0) \equiv 0$, $\varphi_j(t,x) \in S(1,g_\delta)$ has support where $|x| \lesssim \lambda^{-\delta}$, for δ, $\varepsilon > 0$. If $\delta = (1+2\varepsilon)/3$ and $\varepsilon < 1/7$, then (4.25) has an expansion in $S(\lambda^{1-\varepsilon-j(2-5\varepsilon)/3}, g_\delta)$, $j \geq 0$, and is equal to*

$$\begin{aligned}\exp(-i\lambda(\langle x,\xi_0\rangle + \omega(t,x)))&P^*(t,x,D)(\exp(i\lambda(\langle x,\xi_0\rangle+\omega(t,x)))\varphi(t,x))\\ &\sim (\lambda\partial_t\omega(t,x) - r(t,x,\lambda(\xi_0+\partial_x\omega)))\varphi(t,x)\\ + D_t\varphi(t,x) - \sum_j &\partial_{\xi_j} r(t,x,\lambda(\xi_0+\partial_x\omega))D_{x_j}\varphi(t,x) + q_0(t,x,\lambda(\xi_0+\partial_x\omega))\varphi(t,x)\end{aligned} \tag{4.27}$$

modulu terms that are $\mathcal{O}(\lambda^{(7\varepsilon-1)/3})$.

In Section 6 we shall choose $\varepsilon = \varrho = 1/10$ which gives $\delta = 2/5$, $1-4\varepsilon = 3/5$, $1-\delta-\varepsilon = 1/2$ and $(7\varepsilon-1)/3 = -1/10$.

5. The eikonal equation

The first term in the expansion (4.27) is the eikonal equation

$$\partial_t\omega - s(t,x,\xi_0+\partial_x\omega) = 0 \qquad \omega(0,x) = 0 \tag{5.1}$$

where $s(t,x,\xi) = \lambda^{-1}r(t,x,\lambda\xi)$. This we can solve by using the Hamilton-Jacobi equations:

$$\begin{cases} \partial_t x = -\partial_\xi s(t,x,\xi_0+\xi) \\ \partial_t\xi = \partial_x s(t,x,\xi_0+\xi) \end{cases} \tag{5.2}$$

with $(x(0),\xi(0)) = (x,0)$, and letting $\partial_x\omega = \xi$ and $\partial_t\omega = s(t,x,\xi_0+\partial_x\omega)$ with $\omega(0,x) = 0$. Since $s = \nabla s = 0$ on Γ we find that $\partial_t x = \partial_t\xi = 0$ when $x = \xi = 0$, so by uniqueness $\partial_t\omega(t,0) \equiv \partial_x\omega(t,0) \equiv 0$.

We shall solve the Hamilton-Jacobi equations by scaling. Recall that $s(t,x,\xi) \in S(\lambda^{-\varepsilon}, g_\varepsilon)$ for some chosen $0 < \varepsilon < 1/3$ in homogeneous coordinates by Proposition 4.3. By Proposition 4.4 we may assume that $L(t) \equiv \{\,(t,x,0,0)\,\}$, $\forall\, t$, and $\partial_x^2 s = 0$ on Γ. Since s, ∂s and $\partial_x^2 s$ vanish on Γ, Taylor's formula gives

$$\partial_\xi s(t,x,\xi_0+\xi) = \partial_x\partial_\xi s(t,0,\xi_0)x + \partial_\xi^2 s(t,0,\xi_0)\xi + \langle \varrho_1(t,x,\xi)w,w\rangle \tag{5.3}$$

where $w = (x, \xi)$, $\partial_x \partial_\xi s(t, 0, \xi_0) = \mathcal{O}(1)$ by (2.5), $\partial_\xi^2 s(t, 0, \xi_0) = \mathcal{O}(\lambda^\varepsilon)$ and $\varrho_1 \in S(\lambda^{2\varepsilon}, g_\varepsilon)$, since $\widetilde{p}(t, x; \tau, \xi) \cong \tau - r(t, x, \xi)$ modulo terms vanishing of third order at Γ. Similarly, we find

$$\partial_x s(t, x, \xi_0 + \xi) = \partial_x \partial_\xi s(t, 0, \xi_0)\xi + \langle \varrho_2(t, x, \xi)w, w\rangle \tag{5.4}$$

where $\varrho_2 \in S(\lambda^{2\varepsilon}, g_\varepsilon)$.

Now we put $(x, \xi) = (y\lambda^{-3\varepsilon}, \eta\lambda^{-4\varepsilon})$. Then by using (5.3) and (5.4) we find that (5.2) transforms into

$$\begin{cases} \partial_t y = -B(t)y - C(t)\eta + \sigma_1(t, z) \\ \partial_t \eta = B(t)\eta + \sigma_2(t, z) \end{cases} \tag{5.5}$$

where $z = (y, \eta)$, $B(t) = \partial_x \partial_\xi s(t, 0, \xi_0)$ and $C(t) = \lambda^{-\varepsilon}\partial_\xi^2 s(t, 0, \xi_0)$ are uniformly bounded, and

$$\begin{cases} \sigma_1(t, z) = \lambda^{-3\varepsilon}\langle \varrho_1(t, y\lambda^{-3\varepsilon}, \eta\lambda^{-4\varepsilon})(y, \lambda^{-\varepsilon}\eta), (y, \lambda^{-\varepsilon}\eta)\rangle \\ \sigma_2(t, z) = -\lambda^{-2\varepsilon}\langle \varrho_2(t, y\lambda^{-3\varepsilon}, \eta\lambda^{-4\varepsilon})(y, \lambda^{-\varepsilon}\eta), (y, \lambda^{-\varepsilon}\eta)\rangle \end{cases} \tag{5.6}$$

are uniformly bounded in C^∞ and vanish of second order in z. Then (5.5) has a uniformly bounded C^∞ solution if $z(0)$ is uniformly bounded. This means that if $x(0) = \mathcal{O}(\lambda^{-3\varepsilon})$ and $\xi(0) = 0$ then we find $x = \mathcal{O}(\lambda^{-3\varepsilon})$ and $\partial_x \omega = \xi = \mathcal{O}(\lambda^{-4\varepsilon})$ for any $t \in I$. The scaling gives that $\partial_x^\alpha \partial_x \omega = \mathcal{O}(\lambda^{(-4+3|\alpha|)\varepsilon})$ when $|x| \lesssim \lambda^{-3\varepsilon}$. Since $\omega(t, 0) \equiv 0$ we obtain that $\omega = \mathcal{O}(\lambda^{-7\varepsilon})$ when $|x| \lesssim \lambda^{-3\varepsilon}$, thus $\omega(t, \cdot) \in S(\lambda^{-7\varepsilon}, g_{3\varepsilon})$.

Now by differentiating (5.1) twice we find that

$$\partial_t \partial_x^2 \omega(t, 0) = 2\operatorname{Re}\left(\partial_\xi \partial_x s(t, 0, \xi_0)\partial_x^2 \omega(t, 0)\right) + \partial_x^2 \omega(t, 0)\partial_\xi^2 s(t, 0, \xi_0)\partial_x^2 \omega(t, 0)$$

because $\partial_x \omega(t, 0) = \partial_\xi s(t, 0, \xi_0) = \partial_x^2 s(t, 0, \xi_0) = 0$. Since $\partial_x^2 \omega(0, x) \equiv 0$ we find by uniqueness that $\partial_x^2 \omega(t, 0) \equiv 0$. This gives that $\partial_t \omega = \mathcal{O}(\lambda^{-7\varepsilon})$ when $|x| \lesssim \lambda^{-3\varepsilon}$, and we obtain the following result.

Proposition 5.1. *Let $0 < \varepsilon < 1/3$, and assume that Propositions 4.3 and 4.4 hold. Then there exists a real $\omega(t, \cdot) \in S(\lambda^{-7\varepsilon}, g_{3\varepsilon})$ satisfying $\partial_t \omega = \lambda^{-1} r(t, x, \lambda(\xi_0 + \partial_x \omega))$ when $|x| \lesssim \lambda^{-3\varepsilon}$ and $t \in I$ such that $\omega(t, 0) \equiv 0$ and $\partial_x^2 \omega(t, 0) \equiv 0$. We find that the values of $(t, x; \lambda\partial_t \omega(t, x), \lambda(\xi_0 + \partial_x \omega(t, x)))$ have homogeneous distance to the rays through Γ which is $\lesssim \lambda^{-3\varepsilon}$ when $|x| \lesssim \lambda^{-3\varepsilon}$ and $t \in I$.*

6. The transport equations

The next term in (4.27) is the transport equation, which is equal to

$$D_p \varphi_0 + q_0 \varphi_0 = 0 \qquad \text{at } \Gamma \tag{6.1}$$

where $D_p = D_t - \sum_j \partial_{\xi_j} r(t, x, \lambda(\xi_0 + \partial_x \omega(t, x)))D_{x_j} = D_t$ when $x = 0$ and

$$q_0(t) = D_t |\nabla p(t, 0, \xi_0)|/2|\nabla p(t, 0, \xi_0)| + p_0(t, 0, \xi_0)/|\nabla p(t, 0, \xi_0)| = \mathcal{O}(\lambda^\varepsilon) \tag{6.2}$$

modulo $\mathcal{O}(\lambda^{2\varepsilon}|x|)$ when $|x| \lesssim \lambda^{-\delta}$ by (4.18). Here $\omega(t, x)$ is given by Proposition 5.1.

Lemma 6.1. *We have that*

$$D_p = D_t + \sum_j \langle a_j(t), x\rangle D_{x_j} + R(t,x,D_x)$$

where $\mathbf{R}^{n-1} \ni a_j(t) = \mathcal{O}(1)$ *and* $R(t,x,D_x)$ *is a first order differential operator in* x *with coefficients that are* $\mathcal{O}(\lambda^{3\varepsilon}|x|^2)$ *when* $|x| \lesssim \lambda^{-3\varepsilon}$.

Proof. As before, we shall put $s(t,x,\xi) = \lambda^{-1}r(t,x,\lambda\xi) \in S(\lambda^{-\varepsilon}, g_\varepsilon)$. Since $\partial_x^2\omega(t,0) \equiv 0$ by Proposition 5.1 we have from Taylor's formula that $a_j(t) = -\partial_x\partial_{\xi_j}s(t,0,\xi_0)$ which is uniformly bounded by (4.13) and Proposition 4.4. The coefficients of the error term R are given by the second order x derivatives of the coefficients of D_p which are

$$\partial_x^2\partial_\xi s + 2\,\mathrm{Re}\left(\partial_x\partial_\xi^2 s\partial_x^2\omega\right) + \partial_x^2\omega\partial_\xi^3 s\partial_x^2\omega + \partial_\xi^2 s\partial_x^3\omega = \mathcal{O}(\lambda^{3\varepsilon})$$

when $|x| \lesssim \lambda^{-3\varepsilon} \ll \lambda^{-\varepsilon}$ by Propositions 4.3 and 5.1, which proves the result. □

We obtain new variables y in $\mathbf{R}^{n-1}$ by solving

$$\partial_t y_j = \langle a_j(t), y\rangle \qquad y_j(0) = x_j \qquad \forall\, j$$

Then $D_t+\sum_j\langle a_j(t),x\rangle D_{x_j}$ is changed into D_t but $D_{x_j} = D_{y_j}$ is unchanged, and we will for simplicity keep the notation (t,x). The change of variables is uniformly bounded since $a_j = \mathcal{O}(1)$, so it preserves the neighborhoods $|x| \lesssim \lambda^{-\nu}$ and symbol classes $S(\lambda^\mu, g_\nu)$, $\forall\, \mu,\, \nu$. We shall then solve the approximate transport equation

$$D_t\varphi_0 + q_0(t)\varphi_0 = 0 \tag{6.3}$$

where $\varphi_0(0,x) \in S(1,g_\delta)$ is supported where $|x| \lesssim \lambda^{-\delta}$ and $q_0(t)$ is given by (6.2). If $\lambda^{-\delta} \ll \lambda^{-3\varepsilon}$ then by Lemma 6.1 the approximation errors will be in $S(\lambda^{3\varepsilon-\delta}, g_\delta)$, so we will assume $\delta > 3\varepsilon$. In fact, since ∂_x maps $S(1,g_\delta)$ into $S(\lambda^\delta, g_\delta)$ and $|x| \lesssim \lambda^{-\delta}$, we find $R(t,x,D_x)\varphi_0 \in S(\lambda^{3\varepsilon-\delta}, g_\delta)$. If we put $\delta = 4\varepsilon$ then the approximation errors in the transport equation will be $\mathcal{O}(\lambda^{-\varepsilon})$.

If we choose the initial data $\varphi_0(0,x) = \phi_0(x) = \varphi(\lambda^\delta x)$, where $\varphi \in C_0^\infty$ satisfies $\varphi(0) = 1$, we obtain the solution

$$\varphi_0(t,x) = \phi_0(x)\exp(-iB(t)) \tag{6.4}$$

where $B' = q_0$ and $\mathrm{B}(0) = 0$. By condition (4.2) we find that $|\varphi_0(t,x)| \le |\varphi(\lambda^\delta x)|$, and $|x| \lesssim \lambda^{-\delta}$ in $\operatorname{supp}\varphi_0$, which also holds in the original x coordinates. Observe that $D_x^\alpha\varphi_0 = \mathcal{O}(\lambda^{\delta|\alpha|})$, $\forall\,\alpha$, and we have from the transport equation that $D_t\varphi_0 = -q_0\varphi_0 = \mathcal{O}(\lambda^\varepsilon)$ by (6.2). Since $D_t^k q_0 = \mathcal{O}(\lambda^{\varepsilon(k+1)})$ by Proposition 4.3, we find by induction that $\varphi_0 \in S(1,g_\delta)$.

After solving the eikonal equation and the approximate transport equation, we find from Proposition 4.5 that the terms in the expansion (4.27) are $\mathcal{O}(\lambda^{(7\varepsilon-1)/3}) + \mathcal{O}(\lambda^{-\varepsilon})$, if $\varepsilon < 1/7$ and $\delta = (1+2\varepsilon)/3 = 4\varepsilon$, and all the terms contain the factor $\exp(-iB(t))$. We take $\varepsilon = 1/10$, $\delta = 2/5$ which gives $(7\varepsilon-1)/3 = -\varepsilon = -1/10$. Then the expansion in

Proposition 4.5 is in multiples of $\lambda^{-1/2}$, but since the terms of (4.27) are $\mathcal{O}(\lambda^{-1/10})$ we will take $\varrho = 1/10$.

Thus the approximate transport equation for φ_1 is

$$D_t\varphi_1 + q_0(t)\varphi_1 = \lambda^{1/10}R_1\exp(-iB(t)) \qquad \text{at } \Gamma \tag{6.5}$$

where R_1 is uniformly bounded in the symbol class $S(\lambda^{-1/10}, g_{2/5})$ and supported where $|x| \lesssim \lambda^{-2/5}$. In fact, R_1 contains both the error terms from the transport equation (6.1) for φ_0 and the terms that are $\mathcal{O}(\lambda^{-1/10})$ in (4.27). By putting

$$\varphi_1(t,x) = \exp(-iB(t))\phi_1(t,x)$$

the transport equation reduces to solving

$$D_t\phi_1 = \lambda^{1/10}R_1 \tag{6.6}$$

with initial values $\phi_1(0,x) = 0$. Then $\phi_1 \in S(1, g_{2/5})$ will have support where $|x| \lesssim \lambda^{-2/5}$.

Similarly, the general term in the expansion is $\varphi_k\lambda^{-k/10}$ where φ_k will solve the approximate transport equation

$$D_t\varphi_k + q_0(t)\varphi_k = \lambda^{k/10}R_k\exp(iB(t)) \qquad k \ge 1 \tag{6.7}$$

with R_k is uniformly bounded in the symbol class $S(\lambda^{-k/10}, g_{2/5})$ and is supported where $|x| \lesssim \lambda^{-2/5}$. In fact, R_k contains the error terms from the transport equation (6.1) for φ_{k-1} and also the terms that are $\mathcal{O}(\lambda^{-k/10})$ in (4.27). Taking $\varphi_k = \exp(-iB(t))\phi_k$ we obtain the equation

$$D_t\phi_k = \lambda^{k/10}R_k \in S(1, g_{2/5}) \tag{6.8}$$

with initial values $\phi_k(0,x) = 0$, which can be solved with $\phi_k \in S(1, g_{2/5})$ uniformly having support where $|x| \lesssim \lambda^{-2/5}$. Proceeding we obtain an solution modulo $\mathcal{O}(\lambda^{-N/10})$ for any N.

Proposition 6.2. *Choosing $\delta = 2/5$, $\varepsilon = 1/10$ and $\varrho = 1/10$ we can solve the transport equations* (6.1) *and* (6.7) *with $\varphi_k \in S(1, g_{2/5})$ uniformly having support where $|x| \lesssim \lambda^{-2/5}$ and $|t| \lesssim 1$, $\forall\, k$, such that $\varphi_0(0,0) = 1$ and $\varphi_k(0,x) \equiv 0$, $k \ge 1$.*

Now, we get localization in x from the initial values and the transport equation. To get localization in t we use that $\operatorname{Im} B(t) \le 0$. Then we find that $\operatorname{Re}(-iB) \le 0$ with equality at $t = 0$. Near $\partial\Gamma$ we may assume that $\operatorname{Re}(-iB(t)) \ll -\log\lambda$ in an interval of length $\mathcal{O}(\lambda^{-\varepsilon}) = \mathcal{O}(\lambda^{-1/10})$ by (4.3), (4.4) and (4.6). Thus by applying a cut-off function

$$\chi(t) \in S(1, \lambda^{1/5}dt^2) \subset S(1, g_{2/5})$$

such that $\chi(0) = 1$ and $\chi'(t)$ is supported where (4.5) holds, i.e., where $\varphi_k = \mathcal{O}(\lambda^{-N})$, $\forall\, k$, we obtain a solution modulo $\mathcal{O}(\lambda^{-N})$ for any N. In fact, if u_λ is defined by (4.23) and Q by Proposition 4.3 then

$$Q\chi u_\lambda = \chi Q u_\lambda + [Q, \chi]u_\lambda$$

where $[Q,\chi] = D_t\chi$ is supported where $u_\lambda = \mathcal{O}(\lambda^{-N})$ which gives terms that are $\mathcal{O}(\lambda^{-N})$, $\forall\, N$. Thus, by solving the eikonal equation (5.1) for ω and the transport equations (6.7) for φ_k for $k \le 10N$, we obtain that $Q\chi u_\lambda = \mathcal{O}(\lambda^{-N})$ for any N and we get the following remark.

Remark 6.3. In Proposition 6.2 we may assume that $\varphi_k(t,x) = \phi_k(\lambda^{1/10}t, \lambda^{2/5}x) \in S(1, g_{2/5})$, $k \ge 0$, where $\phi_k \in C_0^\infty$ has support where $|x| \lesssim 1$ and $|t| \lesssim \lambda^{1/10} \le \lambda^{2/5}$, $\lambda \ge 1$.

7. The proof of Theorem 2.9

For the proof we will need the following modification of [4, Lemma 26.4.14]. Recall that $\mathcal{D}'_\Gamma = \{\, u \in \mathcal{D}' : \mathrm{WF}(u) \subset \Gamma \,\}$ for $\Gamma \subset T^*\mathbf{R}^n$, and that $\|u\|_{(k)}$ is the L^2 Sobolev norm of order k of $u \in C_0^\infty$.

Lemma 7.1. *Let*

$$u_\lambda(x) = \lambda^{(n-1)\delta/2} \exp(i\lambda^\varrho \omega(\lambda^\varepsilon x)) \sum_{j=0}^{M} \varphi_j(\lambda^\delta x)\lambda^{-j\kappa} \tag{7.1}$$

with $\omega \in C^\infty(\mathbf{R}^n)$ satisfying $\operatorname{Im}\omega \ge 0$ and $|d\omega| \ge c > 0$, $\varphi_j \in C_0^\infty(\mathbf{R}^n)$, $\lambda \ge 1$, ε, δ, κ and ϱ are positive such that $\varepsilon < \delta < \varepsilon + \varrho$. Here ω and φ_j may depend on λ but uniformly, and φ_j has fixed compact support in all but one of the variables, for which the support is bounded by $C\lambda^\delta$. Then for any integer N we have

$$\|u_\lambda\|_{(-N)} \le C\lambda^{-N(\varepsilon+\varrho)} \tag{7.2}$$

If $\varphi_0(x_0) \ne 0$ and $\operatorname{Im}\omega(x_0) = 0$ for some x_0 then there exists $c > 0$ and $\lambda_0 \ge 1$ so that

$$\|u_\lambda\|_{(-N)} \ge c\lambda^{-(N+\frac{n}{2})(\varepsilon+\varrho)+(n-1)\delta/2} \qquad \lambda \ge \lambda_0 \tag{7.3}$$

Let $\Sigma = \bigcap_{\lambda \ge 1} \bigcup_j \operatorname{supp} \varphi_j(\lambda\, \cdot)$ and let Γ be the cone generated by

$$\{\, (x, \partial\omega(x)),\ x \in \Sigma,\ \operatorname{Im}\omega(x) = 0 \,\} \tag{7.4}$$

then for any real m we find $\lambda^m u_\lambda \to 0$ in $\mathcal{D}'_\Gamma$ so $\lambda^m A u_\lambda \to 0$ in C^∞ if A is a pseudo-differential operator such that $\mathrm{WF}(A) \cap \Gamma = \emptyset$. The estimates are uniform if $\omega \in C^\infty$ uniformly with fixed lower bound on $|d \operatorname{Re}\omega|$, and $\varphi_j \in C^\infty$ uniformly.

We shall use Lemma 7.1 for u_λ in (4.23), then ω will be real valued and Γ in (7.4) will be the bicharacteristic Γ_j converging to a limit bicharacteristic.

Proof of Lemma 7.1. We shall adapt the proof of [4, Lemma 26.4.14] to this case. By making the change of variables $y = \lambda^\varepsilon x$ we find that

$$\hat{u}_\lambda(\xi) = \lambda^{(n-1)\delta/2 - n\varepsilon} \sum_{j=0}^{M} \lambda^{-j\kappa} \int e^{i(\lambda^\varrho\omega(y) - \langle y, \xi/\lambda^\varepsilon\rangle)} \varphi_j(\lambda^{\delta-\varepsilon} y)\, dy \tag{7.5}$$

Let U be a neighborhood of the projection on the second component of the set in (7.4). When $\xi/\lambda^{\varepsilon+\varrho} \notin U$ then for $\lambda \gg 1$ we have that

$$\bigcup_j \operatorname{supp} \varphi_j(\lambda^{\delta-\varepsilon}\cdot) \ni y \mapsto (\lambda^\varrho \omega(y) - \langle y, \xi/\lambda^\varepsilon\rangle)/(\lambda^\varrho + |\xi|/\lambda^\varepsilon)$$
$$= (\omega(y) - \langle y, \xi/\lambda^{\varepsilon+\varrho}\rangle)/(1 + |\xi|/\lambda^{\varepsilon+\varrho})$$

is in a compact set of functions with non-negative imaginary part with a fixed lower bound on the gradient of the real part. Thus, by integrating by part in (7.5) we find for any positive integer m that

$$|\hat{u}_\lambda(\xi)| \le C_m \lambda^{-(n-1)\delta/2 + m(\delta-\varepsilon)}(\lambda^\varrho + |\xi|/\lambda^\varepsilon)^{-m} \qquad \xi/\lambda^{\varepsilon+\varrho} \notin U \qquad \lambda \gg 1 \tag{7.6}$$

This gives any negative power of λ for m large enough since $\delta < \varepsilon + \varrho$. If V is bounded and $0 \notin \overline{V}$ then since u_λ is uniformly bounded in L^2 we find

$$\int_{\tau V} |\hat{u}_\lambda(\xi)|^2 (1 + |\xi|^2)^{-N}\, d\xi \le C_V \tau^{-2N} \qquad \tau \ge 1$$

Using this estimate with $\tau = \lambda^{\varepsilon+\varrho}$ together with the estimate (7.6) we obtain (7.2). If $\chi \in C_0^\infty$ then we may apply (7.6) to χu_λ, thus we find for any positive integer j that

$$|\widehat{\chi u}_\lambda(\xi)| \le C_j \lambda^{-(n-1)\delta/2 + j(\delta-\varepsilon)}(\lambda^\varrho + |\xi|/\lambda^\varepsilon)^{-j} \qquad \xi \in W \qquad \lambda \gg 1$$

if W is any closed cone with $\Gamma \bigcap (\operatorname{supp} \chi \times W) = \emptyset$. Thus we find that $\lambda^m u_\lambda \to 0$ in $\mathcal{D}'_\Gamma$ for every m. To prove (7.3) we assume $x_0 = 0$ and take $\psi \in C_0^\infty$. If $\operatorname{Im} \omega(0) = 0$ and $\varphi(0) \ne 0$ we find

$$\lambda^{n(\varepsilon+\varrho)-(n-1)\delta/2} e^{-i\lambda^\varrho \operatorname{Re} \omega(0)} \langle u_\lambda, \psi(\lambda^{\varepsilon+\varrho}\cdot)\rangle$$
$$= \int e^{i\lambda^\varrho(\omega(x/\lambda^\varrho) - \omega(0))} \psi(x) \sum_j \varphi_j(x/\lambda^{\varepsilon+\varrho-\delta})\lambda^{-j\kappa}\, dx$$
$$\to \int e^{i\langle \operatorname{Re} \partial_x \omega(0), x\rangle} \psi(x) \varphi_0(0)\, dx \qquad \lambda \to +\infty$$

which is not equal to zero for some suitable $\psi \in C_0^\infty$. In fact, we have $\varphi_j(x/\lambda^{\varepsilon+\varrho-\delta}) = \varphi_j(0) + \mathcal{O}(\lambda^{\delta-\varepsilon-\varrho}) \to \varphi_j(0)$ when $\lambda \to \infty$, because $\delta < \varepsilon + \varrho$. Since

$$\|\psi(\lambda^{\varepsilon+\varrho}\cdot)\|_{(N)} \le C\lambda^{(N-n/2)(\varepsilon+\varrho)}$$

we obtain that $0 < c \le \lambda^{(N+\frac{n}{2})(\varepsilon+\varrho)-(n-1)\delta/2}\|u\|_{(-N)}$ which gives (7.3) and the lemma. $\square$

Proof of Theorem 2.9. Assume that Γ is a limit bicharacteristic of P. We are going to show that (2.12) does not hold for any ν, N and any pseudodifferential operator A such that $\Gamma \cap \mathrm{WF}(A) = \emptyset$. This means that there exists approximate solutions $0 \ne u_j \in C_0^\infty$ to $P^* u_j \cong 0$ such that

$$\|u_j\|_{(-N)}/(\|P^*u_j\|_{(\nu)} + \|u_j\|_{(-N-n)} + \|Au_j\|_{(0)}) \to \infty \qquad \text{when } j \to \infty \tag{7.7}$$

which will contradict the local solvability of P at Γ by Remark 2.10.

Let Γ_j be a sequence of bicharacteristics of p that converges to $\Gamma \subset \Sigma_2$ and λ_j given by (4.3) and (4.4) with ε to be determined later. Now the conditions and conclusions are invariant under symplectic changes of homogeneous coordinates and multiplication by elliptic pseudodifferential operators. Thus by Proposition 4.3 we may assume that the coordinates are chosen so that $\Gamma_j = I \times (0, 0, \xi_j)$ with $|\xi_j| = 1$, and for any $0 < \varepsilon < 1/3$ and $c > 0$ we can write $P^* = Q + R$ where $\Gamma_j \cap \mathrm{WF}_\varepsilon(R) = \emptyset$ and Q has symbol

$$\tau - r(t, x, \xi) + q_0(t, x, \xi) + r_0(t, x, \xi) \tag{7.8}$$

when $|\xi| \ge c\lambda_j$ and the homogeneous distance to Γ_j is less than $c|\xi|^{-\varepsilon}$. We have that $R \in S^{1+\varepsilon}_{1-\varepsilon,\varepsilon}$, $r_0 \in S^{3\varepsilon-1}_{1-\varepsilon,\varepsilon}$, $q_0 \in S^{\varepsilon}_{1-\varepsilon,\varepsilon}$ is given by (4.18), and $r \in S^{1-\varepsilon}_{1-\varepsilon,\varepsilon}$ vanishes of second order at Γ_j.

Now, we may replace the norms $\|u\|_{(s)}$ in (7.7) by the norms

$$\|u\|_s^2 = \|\langle D_x\rangle^s u\|^2 = \int \langle\xi\rangle^{2s} |\hat{u}(\tau, \xi)|^2 \, d\tau d\xi$$

In fact, the quotient $\langle\xi\rangle / \langle(\tau, \xi)\rangle \cong 1$ when $|\tau| \lesssim |\xi|$, thus in a conical neighborhood of Γ. So replacing the norms in the estimate (7.7) only changes the constant and the operator A in the estimate (2.12). By using Proposition 4.4 we may assume that the grazing Lagrangean space $L_j(w) \equiv \{\, (s, y; 0, 0) : \ (s, y) \in \mathbf{R}^n \,\}$, $\forall\, w \in \Gamma_j$, after conjugation with a uniformly bounded C^1 section $F(t)$ of Fourier integral operators, then $\partial_x^2 r = 0$ at Γ_j and $\Gamma \cap \mathrm{WF}_\varepsilon(A) = \emptyset$. Observe that for each t we find that $F(t)$ is uniformly continuous in local H_s spaces, which we may use in (7.7) after changing A. Also the conjugation of $F(t)$ with the operator with symbol (7.8) gives a uniformly bounded expansion. By changing A again, we may then replace the local $\|u\|_s$ norms by the norms $\|u\|_{(s)}$ in (7.7) so that we can use Lemma 7.1.

Now, by choosing $\delta = 2/5$, $\varepsilon = 1/10$ and $\varrho = 1/10$ and using Propositions 4.5, 5.1, 6.2 and Remark 6.3, we can for each Γ_j construct approximate solution u_{λ_j} on the form (4.23) so that $Qu_{\lambda_j} = \mathcal{O}(\lambda^k)$, for any k. The real valued phase function is equal to is $\langle x, \xi_j\rangle + \omega_j(t, x)$ where $|\xi_j| = 1$ and $\omega_j(t, x) \in S({\lambda_j}^{-7/10}, g_{3/10})$ and the values of $(t, x; \lambda\partial_t\omega_j(t, x), \lambda(\xi_j + \partial_x\omega_j(t, x)))$ have homogeneous distance to the rays through Γ_j which is $\lesssim \lambda^{-2/5}$ when $|x| \lesssim {\lambda_j}^{-2/5}$, i.e., in $\operatorname{supp} u_{\lambda_j}$. Observe that if $\lambda \gg 1$ then we have that $|\xi_0 + \partial_x\omega(t, x)| \ge c$ in $\operatorname{supp} u_{\lambda_j}$ for some $c > 0$. We find that

$$\omega_j(t, x) = {\lambda_j}^{-7/10}\widetilde{\omega}_j{\lambda_j}^{3/10}t, {\lambda_j}^{3/10}x)$$

where $\widetilde{\omega}_j \in C^\infty$ uniformly so $\partial_x\omega = \mathcal{O}(\lambda_j^{-2/5})$ when $x = \mathcal{O}(\lambda_j^{-2/5})$ and

$$\lambda_j(\langle x, \xi_j\rangle + \omega_j(t, x)) = {\lambda_j}^{7/10}\langle {\lambda_j}^{3/10}x, \xi_j\rangle + \lambda^{-4/10}\widetilde{\omega}_j({\lambda_j}^{3/10}t, {\lambda_j}^{3/10}x)$$

Thus $\delta = 2/5$, $\varepsilon = 3/10$, $\kappa = 1/10$ and $\varrho = 7/10$ in (7.1) so we find $\varepsilon + \varrho = 1 > \delta > \varepsilon$.

The amplitude functions $\varphi_{k,j}(t, x) = \phi_{k,j}({\lambda_j}^{2/5}t, {\lambda_j}^{2/5}x)$ where $\phi_{k,j} \in C_0^\infty$ uniformly in j with fixed compact support in x, but in t the support is bounded by $C{\lambda_j}^{2/5}$, so u_{λ_j} will satisfy the conditions in Lemma 7.1 uniformly. Clearly differentiation of Qu_{λ_j} can at

most give a factor λ_j since $\varepsilon + \varrho = 1$ and $\delta < 1$. Because of the bound on the support of u_{λ_j} we may obtain that

$$\|Qu_{\lambda_j}\|_{(\nu)} = \mathcal{O}(\lambda_j^{-N-n}) \tag{7.9}$$

for the chosen ν.

If $\mathrm{WF}(A) \bigcap \Gamma = \emptyset$, then we find $\mathrm{WF}(A) \bigcap \Gamma_j = \emptyset$ for large j, so Lemma 7.1 gives $\|Au_{\lambda_j}\|_{(0)} = \mathcal{O}(\lambda_j^{-N-n})$ when $j \to \infty$. Since $x = \mathcal{O}(\lambda_j^{-2/5})$ in $\operatorname{supp} u_{\lambda_j}$, we find that the values of $(t, x; \lambda\partial_t\omega_j(t, x), \lambda(\xi_j + \partial_x\omega_j(t, x)))$ have homogeneous distance to the rays through Γ_j which is $\lesssim \lambda^{-2/5}$ for $x \in \operatorname{supp} u_{\lambda_j}$, and this converges to Γ. Thus, if $R \in S^{11/10}_{9/10,1/10}$ such that $\mathrm{WF}_{1/10}(R) \bigcup \Gamma_j = \emptyset$ then we find from the expansion (4.24) that all the terms of Ru_{λ_j} vanish for large enough λ_j. In fact, since $\lambda_j^{-2/5} \ll \lambda_j^{-1/10}$ for $j \gg 1$, we find for any α and K that

$$\partial^\alpha R(t, x; \lambda_j((0, \xi_j) + \partial_{t,x}\omega_j(t, x))) = O(\lambda_j^{-K})$$

in $\bigcup_k \operatorname{supp} \varphi_{k,j}$. As before, we find that $\|Ru_{\lambda_j}\|_{(\nu)} = \mathcal{O}(\lambda_j^{-N-n})$ by the bound on the support of u_λ, so we obtain from (7.9) that

$$\|P^*u_{\lambda_j}\|_{(\nu)} = \mathcal{O}(\lambda_j^{-N-n}) \tag{7.10}$$

for the chosen ν.

Since $\varepsilon + \varrho = 1$ we also find from Lemma 7.1 that

$$\lambda_j^{-N} = \lambda_j^{-N(\varepsilon+\varrho)} \gtrsim \|u_{\lambda_j}\|_{(-N)} \gtrsim \lambda_j^{-(N+\frac{n}{2})(\varepsilon+\varrho)+(n-1)\delta/2} = \lambda_j^{-N-n/2+(n-1)/5} \geq \lambda_j^{-N-n/2}$$

when $\lambda_j \geq 1$. We obtain that (7.7) holds for $u_j = u_{\lambda_j}$ when $j \to \infty$, so Remark 2.10 gives that P is not solvable at the limit bicharacteristic Γ. $\square$

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